Introduction to Partonic Structure of Hadrons

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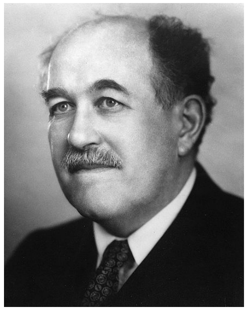
第六届格点量子色动力学夏令营, 2023.07.20

Introduction

- Hadrons (baryons, mesons) are composite particles with quarks and gluons being their fundamental constituents
- First evidence of the composite nature of the proton



Otto Stern

Nobel prize in 1943

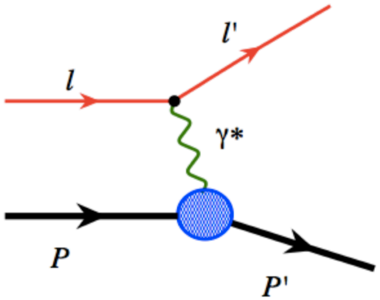
$$\mu_p = g_p \left(\frac{e\hbar}{2m_p} \right)$$

$$g_p = 2.792847356(23) \neq 2!$$

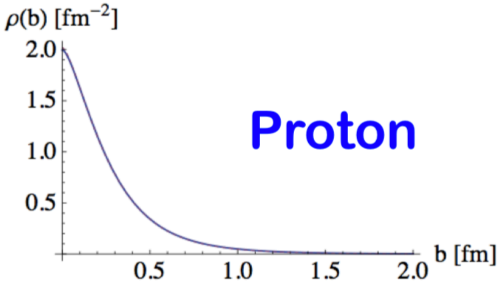
- Elastic e-p scattering maps out the charge and magnetization distribution of the proton

R. Hofstadter

Nobel prize in 1961

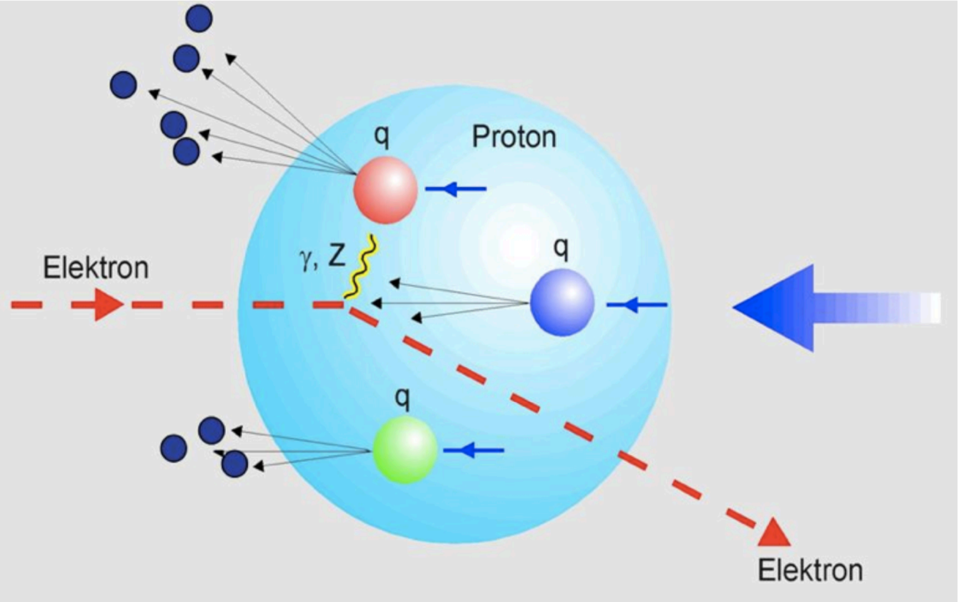


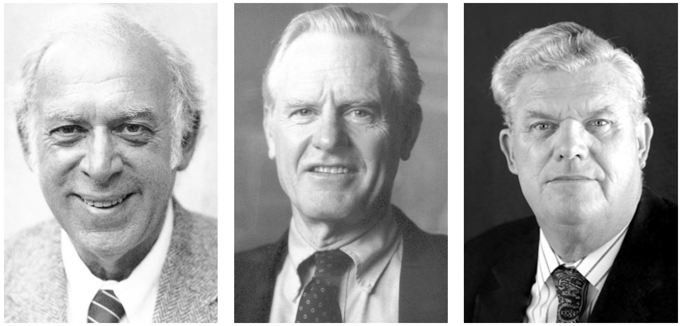
$\Rightarrow$



Introduction

- Hadrons (baryons, mesons) are composite particles with quarks and gluons being their fundamental constituents
- Deep-inelastic scattering accesses the momentum density of the proton's fundamental constituents via knockout reactions





J. Friedman H. Kendall R. Taylor

Nobel prize in 1990

- Discovery of spin-1/2 quarks and partonic structure of the proton

Theoretical tools for hadron structure

- What have we learnt from non-relativistic systems such as atoms?

- A quantum mechanical system is described by its wave function $|\psi\rangle$, which is determined from Schrödinger equation

- Physical observables are usually sensitive to the modulus square of the wave function $|\langle x|\psi\rangle|^2 = |\psi(x)|^2$, where the phase information is washed out

- The complete information of the system can be obtained by measuring correlations of wave functions or the density matrix. For a pure state it is defined as

$$\rho = |\psi\rangle\langle\psi|$$

- In coordinate space, we have

$$\langle x|\rho|x'\rangle = = \langle x|\psi\rangle\langle\psi|x'\rangle = \psi(x)\psi^*(x')$$

Theoretical tools for hadron structure

- The Fourier transform of the density matrix provides an alternative description of a quantum mechanical system. It is called the Wigner function/distribution

$$W(\mathbf{r},\mathbf{p}) = \int \frac{d^3\mathbf{R}}{(2\pi)^3} e^{-i\mathbf{p}\cdot\mathbf{R}} \psi^*\left(\mathbf{r}-\frac{\mathbf{R}}{2}\right)\psi\left(\mathbf{r}+\frac{\mathbf{R}}{2}\right)$$

- It is the quantum analogue of the classical phase-space distribution. It is a real function,

$$W^*(\mathbf{r},\mathbf{p}) = W(\mathbf{r},\mathbf{p})$$

but not positive-definite, and cannot be regarded as a probability distribution

- Nevertheless, physical observables can be computed by taking the average

$$\langle O(\mathbf{r},\mathbf{p})\rangle = \int d^3\mathbf{r}\, d^3\mathbf{p}\, W(\mathbf{r},\mathbf{p}) O(\mathbf{r},\mathbf{p})$$

with the operator being appropriately ordered

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- Integrating over coordinate or momentum does yield positive-definite density functions

$$\int d^3\mathbf{p}\, W(\mathbf{r},\mathbf{p}) = |\psi(\mathbf{r})|^2 = \rho(\mathbf{r}), \qquad \int d^3\mathbf{r}\, W(\mathbf{r},\mathbf{p}) = |\psi(\mathbf{p})|^2 = n(\mathbf{p})$$

- The former represents the spatial distribution of matter (e.g., charge distribution), while the latter represents the density distribution of its constituents in momentum space
- They provide two types of quantities unraveling the microscopic structure of matter

Theoretical tools for hadron structure

- The spatial distribution $\rho(\mathbf{r}) = |\psi(\mathbf{r})|^2$ can be probed through elastic scattering of electrons, photons, etc., off the target, where one measures the elastic form factor $F(\mathbf{\Delta})$ defined as

$$\rho(\mathbf{r}) = \int d^3\mathbf{\Delta} e^{i\mathbf{\Delta}\cdot\mathbf{r}} F(\mathbf{\Delta})$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{point}} |F(\mathbf{\Delta})|^2$$

- The momentum density can be probed through inelastic knockout scattering, where one measures the structure function related to the momentum density

$$n(\mathbf{p}) = \int \frac{d^3\mathbf{r}_1 d^3\mathbf{r}_2}{(2\pi)^6} e^{i\mathbf{p}\cdot(\mathbf{r}_1-\mathbf{r}_2)} \rho(\mathbf{r}_1, \mathbf{r}_2)$$

- These two observables are complementary. The former contains spatial distribution but not velocity information of the constituents, while for the latter it is the opposite

Nucleon form factors

- The spatial distribution and momentum density can be generalized to relativistic systems described by quantum field theory
- Take the nucleon as an example. The spatial distribution can be probed by its elastic form factors. For example, the electromagnetic form factor is given by

$$\langle p_2 | j^\mu(0) | p_1 \rangle = \bar{U}(p_2)\left[\gamma^\mu F_1(\Delta^2) + \frac{i\sigma^{\mu\nu}\Delta_\nu}{2M_N} F_2(\Delta^2)\right] U(p_1), \quad j^\mu(0) = \sum_f Q_f \bar{\psi}_f(0)\gamma^\mu \psi_f(0)$$

- $F_1(\Delta^2), F_2(\Delta^2)$ are called Dirac and Pauli form factors. They are related to the Sachs electric and magnetic form factors by

$$G_E(\Delta^2) = F_1(\Delta^2) - \frac{\Delta^2}{4M_N^2} F_2(\Delta^2), \quad G_M(\Delta^2) = F_1(\Delta^2) + F_2(\Delta^2)$$

which correspond to the charge and magnetization distribution in the Breit frame (the initial and final nucleons have $\mathbf{p}_1 = -\mathbf{p}_2$)

Nucleon form factors

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- This is also reflected from the relation to the charge and magnetic moment of the nucleon

$$Q \equiv \int \mathrm{d}^3 \boldsymbol{r}\, j^0(\boldsymbol{r}), \quad \boldsymbol{\mu} \equiv \int \mathrm{d}^3 \boldsymbol{r} [\boldsymbol{r} \times \boldsymbol{j}(\boldsymbol{r})]$$

$$\frac{\langle p | Q | p \rangle}{\langle p | p \rangle} = F_1(0), \quad \frac{\langle p | \boldsymbol{\mu} | p \rangle}{\langle p | p \rangle} = \frac{\boldsymbol{s}}{M_N}(F_1(0) + F_2(0))$$

- One can sandwich different current operators in the nucleon state, yielding different information about the nucleon structure

Nucleon form factors

- In particular, the axial-vector current helps to reveal the nucleon spin structure

$$\langle p_2 | A^\mu(0) | p_1 \rangle = \bar{U}(p_2)[\gamma^\mu \gamma_5 G_A(\Delta^2) + \frac{\gamma_5 \Delta^\mu}{2M_N} G_P(\Delta^2)] U(p_1)$$

$$A^\mu(0) = \bar{\psi}_f(0)\gamma^\mu \gamma_5 \psi_f(0)$$

- $G_A(\Delta^2), G_P(\Delta^2)$ are the axial and (induced) pseudoscalar form factor
- In analogy with the vector case, the axial charge is defined as the zero momentum transfer limit of $G_A(\Delta^2)$

$$g_A = G_A(\Delta^2 = 0)$$

- The isovector combination g_A^{u-d} is an important parameter dictating the strength of weak interactions of nucleons
- It can be well determined in neutron beta decay experiments
- Ideal for benchmark lattice calculations of nucleon structure
- Disconnected contributions cancel

Nucleon form factors from lattice QCD

- The only systematically improvable tool to study nonperturbative phenomena of hadrons is lattice QCD

- Calculate physical observables from the path integral

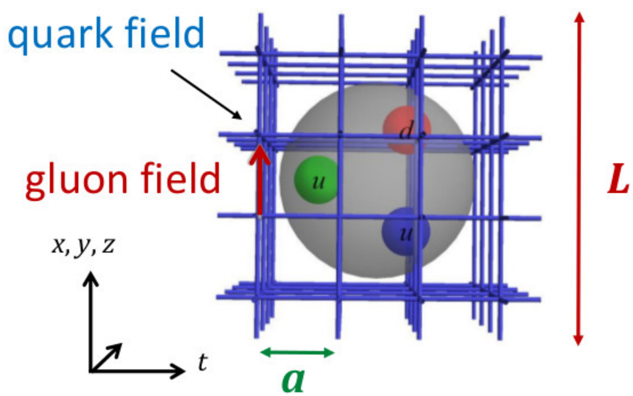
$$\langle 0 | O(\bar{\psi}, \psi, A) | 0 \rangle = \frac{1}{Z} \int \mathscr{D}A \mathscr{D}\bar{\psi} \mathscr{D}\psi \, e^{iS(\bar{\psi},\psi,A)} O(\bar{\psi}, \psi, A)$$

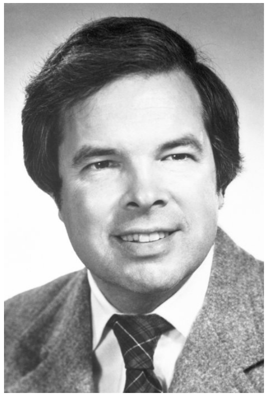
in Euclidean space

$t \to -i\tau, \quad e^{iS_M} \to e^{-S_E}$

- Recover physical limit

$m_\pi \to m_\pi^{phys}, \; a \to 0, \; L \to \infty$





Nucleon form factors from lattice QCD

- Lattice calculation of nucleon axial charge:
- Consider the nucleon 2- and 3-point correlation functions at zero momentum (Fourier transform factors reduce to 1)

$$\mathbf{C}^{\mathrm{2pt}}_{\alpha\beta}(t) = \sum_{\mathbf{x}} \langle 0|\chi_\alpha(t,\mathbf{x})\overline{\chi}_\beta(0,\mathbf{0})|0\rangle \,,$$

$$\mathbf{C}^{\mathrm{3pt}}_{\Gamma;\alpha\beta}(t,\tau) = \sum_{\mathbf{x},\mathbf{x}'} \langle 0|\chi_\alpha(t,\mathbf{x})\mathcal{O}_\Gamma(\tau,\mathbf{x}')\overline{\chi}_\beta(0,\mathbf{0})|0\rangle$$

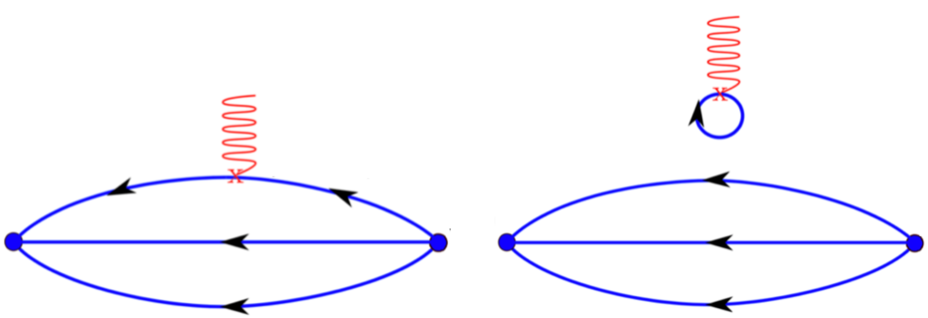
$$\mathcal{O}_\Gamma(x) = \bar{q}(x)\Gamma q(x)$$

$$\Gamma = \gamma_i\gamma_5$$

- with the nucleon interpolating operator

$$\chi(x) = \epsilon^{abc}\left[q_1^{a\,T}(x) C\gamma_5 \frac{(1\pm\gamma_4)}{2} q_2^b(x)\right] q_1^c(x)$$

For a given flavor, quark contraction yields



Connected Disconnected

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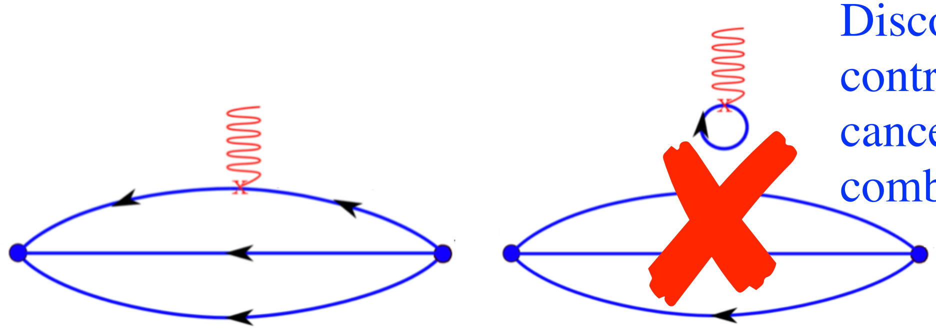
Disconnected contributions cancel for u-d combination



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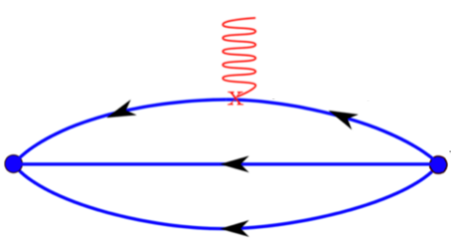
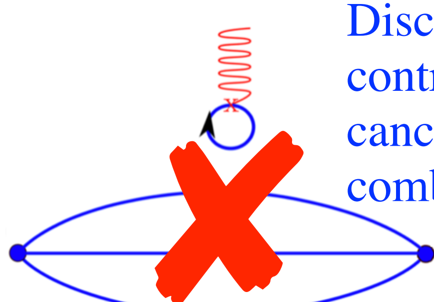


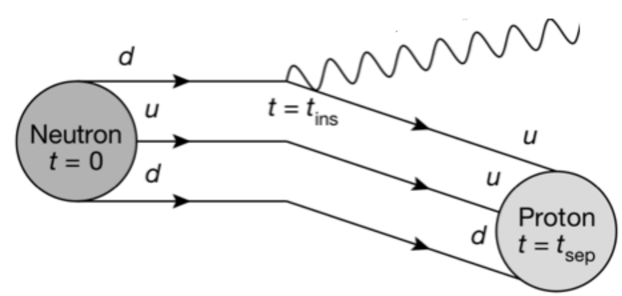
Connected



Disconnected

Disconnected contributions cancel for u-d combination

Equivalent to



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$$\mathcal{O}_\Gamma(x) = \bar{q}(x)\Gamma q(x)$$

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- The nucleon charge is given by

$$\langle N(p,s)|\mathcal{O}^q_\Gamma|N(p,s)\rangle = g^q_\Gamma \bar{u}_s(p)\Gamma u_s(p)$$

$$\sum_s u_s(\mathbf{p})\bar{u}_s(\mathbf{p}) = \not{p} + m_N$$

- To extract the charge, we need the projected correlation functions

$$C^{2\mathrm{pt}}(t) = \langle \mathrm{Tr}[\mathcal{P}_{2\mathrm{pt}}\mathbf{C}^{2\mathrm{pt}}(t)]\rangle$$

$$C^{3\mathrm{pt}}_\Gamma(t,\tau) = \langle \mathrm{Tr}[\mathcal{P}_{3\mathrm{pt}}\mathbf{C}^{3\mathrm{pt}}_\Gamma(t,\tau)]\rangle$$

$$\mathcal{P}_{2\mathrm{pt}} = (1+\gamma_4)/2$$

$$\mathcal{P}_{3\mathrm{pt}} = \mathcal{P}_{2\mathrm{pt}}(1+i\gamma_5\gamma_3)$$

Nucleon form factors from lattice QCD

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$$\mathcal{O}_\Gamma(x) = \bar{q}(x)\Gamma q(x)$$

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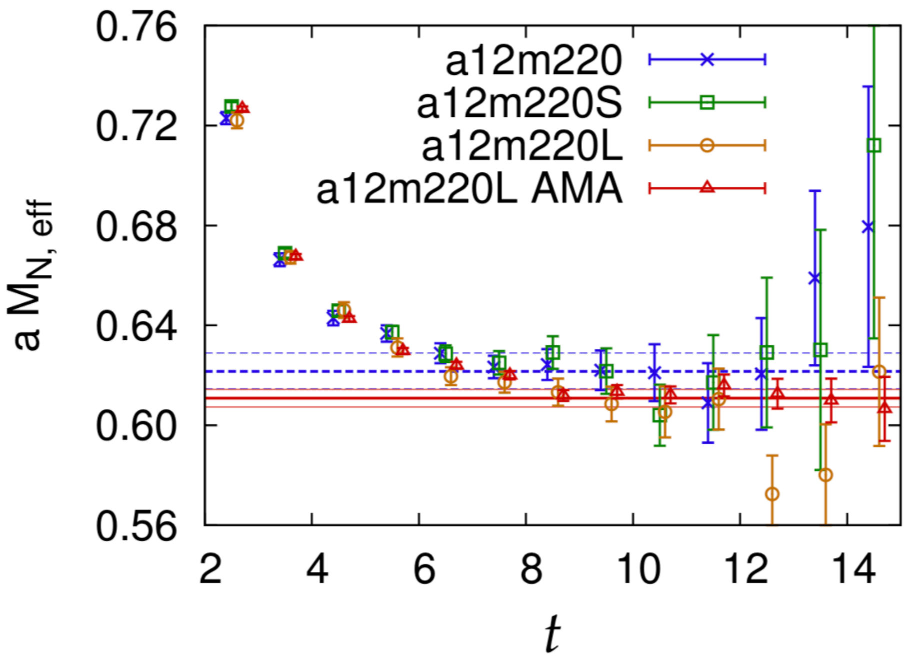
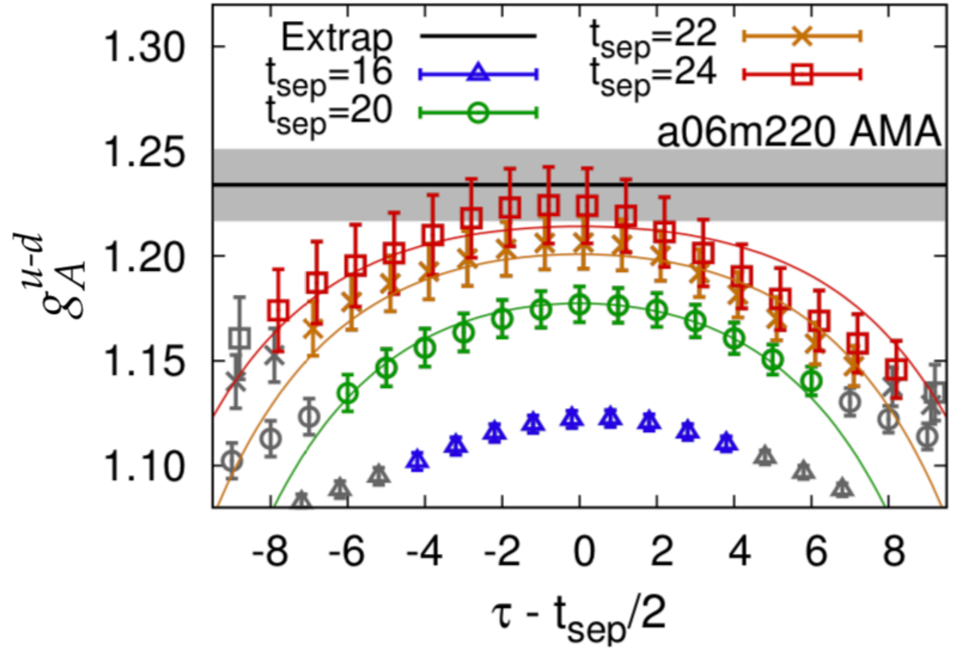
- Two-state fits for the projected 2- and 3-point correlation functions

$$\begin{aligned}
C^{\mathrm{2pt}}(t_f,t_i) = {} & \\
& |\mathcal{A}_0|^2 e^{-M_0(t_f-t_i)} + |\mathcal{A}_1|^2 e^{-M_1(t_f-t_i)}\,,\\
C^{\mathrm{3pt}}_\Gamma(t_f,\tau,t_i) = {} & \\
& |\mathcal{A}_0|^2 \langle 0|\mathcal{O}_\Gamma|0\rangle e^{-M_0(t_f-t_i)} + \\
& |\mathcal{A}_1|^2 \langle 1|\mathcal{O}_\Gamma|1\rangle e^{-M_1(t_f-t_i)} + \\
& \mathcal{A}_0\mathcal{A}_1^* \langle 0|\mathcal{O}_\Gamma|1\rangle e^{-M_0(\tau-t_i)} e^{-M_1(t_f-\tau)} + \\
& \mathcal{A}_0^*\mathcal{A}_1 \langle 1|\mathcal{O}_\Gamma|0\rangle e^{-M_1(\tau-t_i)} e^{-M_0(t_f-\tau)}\,,
\end{aligned}$$

Nucleon form factors from lattice QCD

- Lattice calculation of nucleon axial charge:





Effective mass plot

2-state fit of unrenormalized g_A^{u-d}

Bhattacharya et al, PRD 16’

Nucleon form factors from lattice QCD

- Lattice calculation of nucleon axial charge:
- Renormalization constant

ID	Z_A
a12	0.95(3)
a09	0.95(4)
a06	0.97(3)

Z_A/Z_V
1.045(09)
1.034(11)
1.025(09)

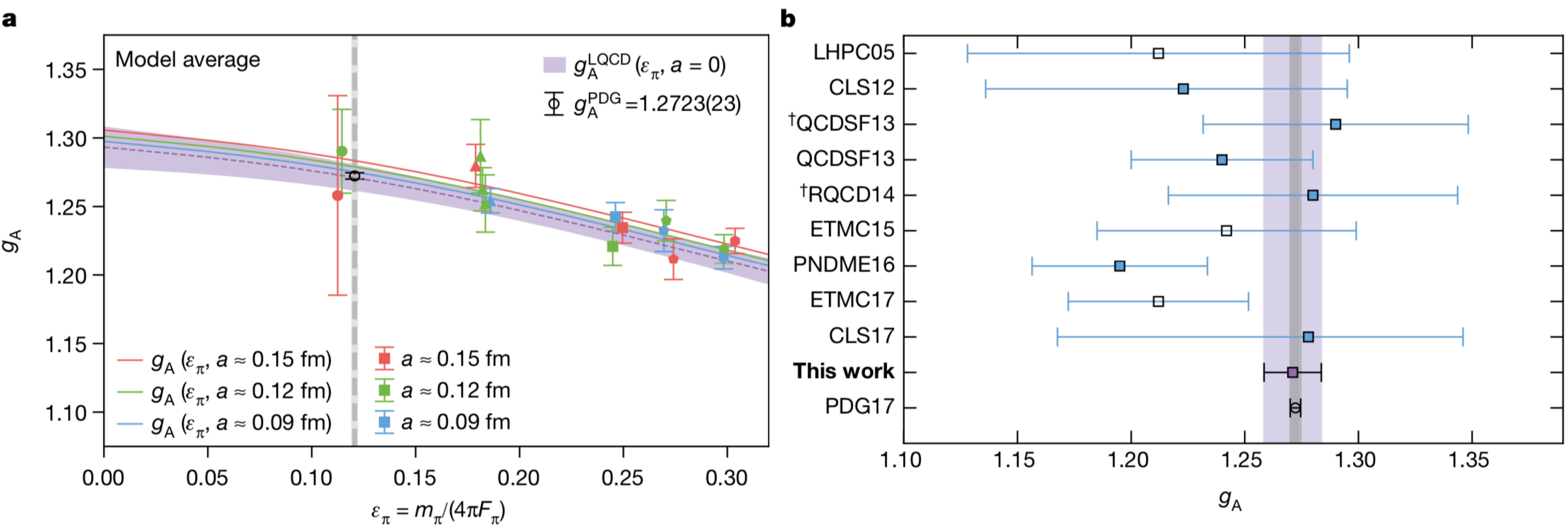
$$Z_V g_V^{u-d} = 1$$

- To compare with experimental measurements, we need to extrapolate to the continuum ($a \to 0$), physical pion mass ($m_\pi = m_{\pi,\text{phys}}$) and the infinite volume limit ($L \to \infty$)

$$g_A^{u-d}(a, m_\pi, L) = c_1 + c_2 a + c_3 m_\pi^2 + c_4 m_\pi^2 e^{-m_\pi L}$$

Nucleon form factors from lattice QCD

- Lattice calculation of nucleon axial charge:

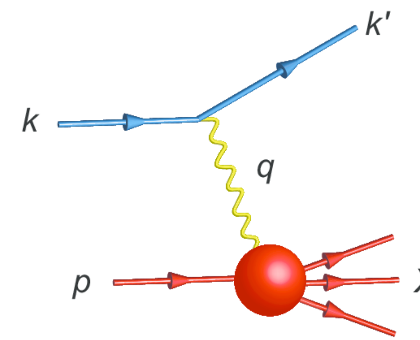


Chang et al, Nature 18'

Parton distribution functions

- Parton distribution functions describe momentum densities of partons inside the nucleon, can be accessed in inclusive DIS
- Scattering amplitude

$$\mathscr{M} = \bar{u}(k')(-ie\gamma_\mu)u(k)\frac{-i}{q^2}\langle X | J^\mu | P\rangle$$



- Differential cross section can be written as

$$\frac{d\sigma}{d\Omega dE} = \frac{\alpha^2}{Q^4}\frac{E'}{E}\ell_{\mu\nu}W^{\mu\nu}$$

- with leptonic and hadronic tensors

$$l_{\mu\nu} = 4e^2(k_\mu k'_\nu + k'_\mu k_\nu - g_{\mu\nu}k\cdot k'), \quad W_{\mu\nu} = \frac{1}{4\pi}\sum_X \langle P | J_\mu | X\rangle\langle X | J_\nu | P\rangle(2\pi)^4\delta^4(P+q-P_X)$$

- General decomposition of $W_{\mu\nu}$ in terms of structure functions

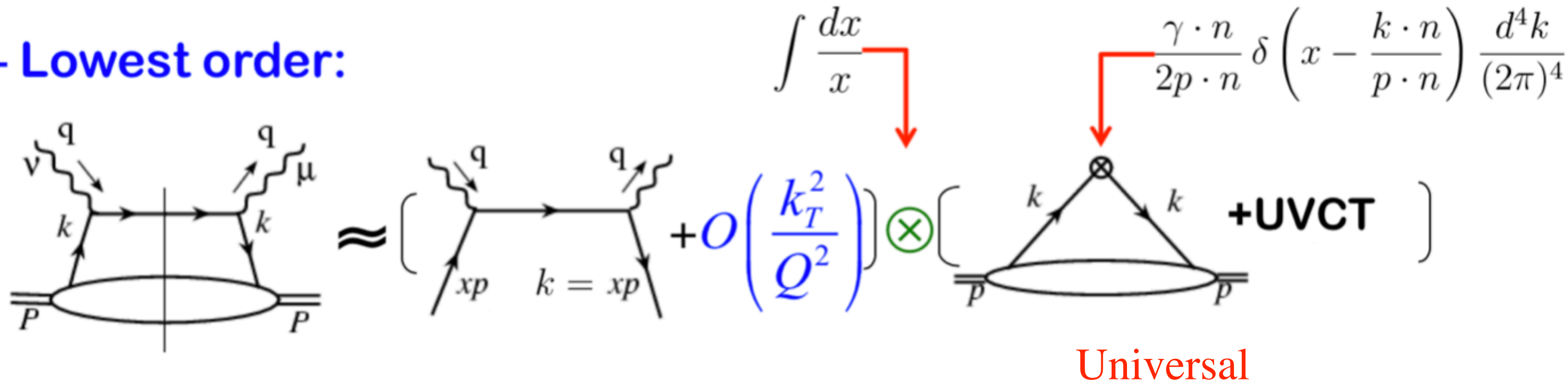
$$W_{\mu\nu} = -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)F_1\left(x_B, Q^2\right) + \frac{1}{p\cdot q}\left(p_\mu - q_\mu\frac{p\cdot q}{q^2}\right)\left(p_\nu - q_\nu\frac{p\cdot q}{q^2}\right)F_2\left(x_B, Q^2\right)$$
$$+iM_p\varepsilon^{\mu\nu\rho\sigma}q_\rho\left[\frac{S_\sigma}{p.q}g_1\left(x_B, Q^2\right) + \frac{(p.q)S_\sigma - (S.q)p_\sigma}{(p.q)^2}g_2\left(x_B, Q^2\right)\right]$$

$$Q^2 = -q^2$$
$$x_B = \frac{Q^2}{2p\cdot q}$$

Parton distribution functions

- Collinear approximation $Q \sim xn \cdot p \gg k_T, \sqrt{k^2}$

– Lowest order:



- A simple example of factorization
- Parton transverse momentum integrated over in the collinear PDFs
- It also provides an estimate of certain power corrections
- In the Bjorken limit $Q^2 \to \infty, x_B$ fixed

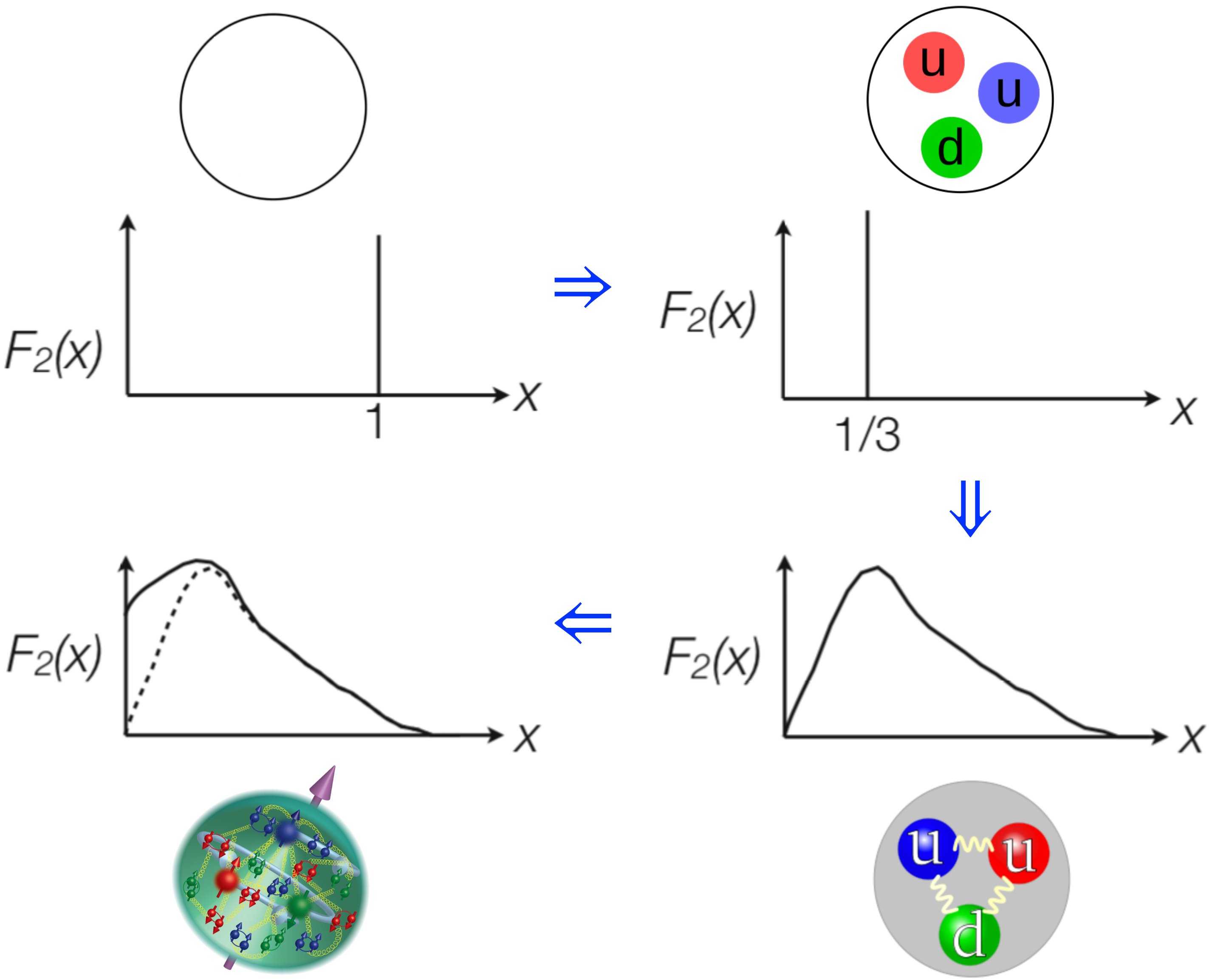
$$F_1(x_B, Q^2) = \frac{1}{2} \sum_i e_i^2 q_i(x_B), \quad F_2(x_B, Q^2) = x_B \sum_i e_i^2 q_i(x_B)$$

up to higher-order perturbative and high-power corrections

- Bjorken scaling and Feynman's parton model

Parton distribution functions

- How do PDFs look like?

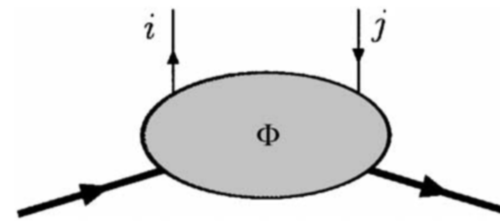


Parton distribution functions

- PDFs from correlation matrix
- Example: The quark PDFs are obtained by applying certain projection to the quark-quark correlation matrix

$$\Phi_{ij}(k,P,S) = \int d^4\xi e^{ik\cdot\xi}\langle PS|\bar{\psi}_j(0)\psi_i(\xi)|PS\rangle$$

$$\mathrm{Tr}(\Gamma\Phi) = \int d^4\xi e^{ik\cdot\xi}\langle PS|\bar{\psi}(0)\Gamma\psi(\xi)|PS\rangle$$



- Not gauge invariant $\psi(x) \to e^{i\alpha(x)}\psi(x)$, $\bar{\psi}(x) \to \bar{\psi}(x)e^{-i\alpha(x)}$
- Needs a gauge link

$$W(x_2,x_1) = \mathscr{P}e^{-ig\int_{x_1}^{x_2} dx\cdot A(x)}, \quad W(x_2,x_1) \to e^{i\alpha(x_2)}W(x_2,x_1)e^{-i\alpha(x_1)}$$

- This correlation matrix satisfies certain constraints from hermiticity, parity and time-reversal invariance

$$\Phi^\dagger(k,P,S) = \gamma^0\Phi(k,P,S)\gamma^0 \qquad \text{Hermiticity}$$

$$\Phi(k,P,S) = \gamma^0\Phi(\tilde{k},\tilde{P},-\tilde{S})\gamma^0 \qquad \text{Parity} \qquad \tilde{k}^\mu = (k^0,-\mathbf{k})$$

$$\Phi^*(k,P,S) = \gamma_5 C\Phi(\tilde{k},\tilde{P},\tilde{S})C^\dagger\gamma_5 \qquad \text{Time reversal}$$

Parton distribution functions

- Φ can be decomposed in terms of Dirac matrices

$$\Phi(k,P,S) = \tfrac{1}{2}\{\mathscr{S}\mathbb{1} + \mathscr{V}_\mu\gamma^\mu + \mathscr{A}_\mu\gamma_5\gamma^\mu + \mathrm{i}\mathscr{P}_5\gamma_5 + \tfrac{1}{2}\,\mathrm{i}\mathscr{T}_{\mu\nu}\sigma^{\mu\nu}\gamma_5\}$$

- with the coefficients of each matrix

$$\mathscr{S} = \tfrac{1}{2}\,\mathrm{Tr}(\Phi) = C_1,$$

$$\mathscr{V}^\mu = \tfrac{1}{2}\,\mathrm{Tr}(\gamma^\mu\Phi) = C_2P^\mu + C_3k^\mu,$$

$$\mathscr{A}^\mu = \tfrac{1}{2}\,\mathrm{Tr}(\gamma^\mu\gamma_5\Phi) = C_4S^\mu + C_5k\cdot SP^\mu + C_6k\cdot Sk^\mu,$$

$$\mathscr{P}_5 = \frac{1}{2\mathrm{i}}\,\mathrm{Tr}(\gamma_5\Phi) = 0,$$

$$\mathscr{T}^{\mu\nu} = \frac{1}{2\mathrm{i}}\,\mathrm{Tr}(\sigma^{\mu\nu}\gamma_5\Phi) = C_7\,P^{[\mu}S^{\nu]} + C_8k^{[\mu}S^{\nu]} + C_9k\cdot SP^{[\mu}k^{\nu]},$$

- $C_i = C_i(k^2, k\cdot P)$ are real functions

Parton distribution functions

- In the collinear approximation

$$k^\mu \approx xP^\mu,\ S^\mu \approx \lambda_N \frac{P^\mu}{M} + S_\perp^\mu$$

- To leading-power accuracy, only three terms are left

$$\mathscr{V}^\mu = \frac{1}{2}\int \mathrm{d}^4\xi\, \mathrm{e}^{ik\cdot\xi} \langle PS|\bar{\psi}(0)\gamma^\mu\psi(\xi)|PS\rangle = A_1 P^\mu,$$

$$\mathscr{A}^\mu = \frac{1}{2}\int \mathrm{d}^4\xi\, \mathrm{e}^{ik\cdot\xi} \langle PS|\bar{\psi}(0)\gamma^\mu\gamma_5\psi(\xi)|PS\rangle = \lambda_N A_2 P^\mu,$$

$$\mathscr{T}^{\mu\nu} = \frac{1}{2\mathrm{i}}\int \mathrm{d}^4\xi\, \mathrm{e}^{ik\cdot\xi} \langle PS|\bar{\psi}(0)\sigma^{\mu\nu}\gamma_5\psi(\xi)|PS\rangle = A_3 P^{[\mu} S_\perp^{\nu]},$$

- and

$$\Phi(k,P,S) = \tfrac{1}{2}\{A_1 \not{P} + A_2\lambda_N\gamma_5\not{P} + A_3\not{P}\gamma_5\not{S}_\perp\}$$

$$A_1 = \frac{1}{2P^+}\mathrm{Tr}(\gamma^+\Phi), \qquad \lambda_N A_2 = \frac{1}{2P^+}\mathrm{Tr}(\gamma^+\gamma_5\Phi), \qquad S_\perp^i A_3 = \frac{1}{2P^+}\mathrm{Tr}(\mathrm{i}\sigma^{i+}\gamma_5\Phi) = \frac{1}{2P^+}\mathrm{Tr}(\gamma^+\gamma^i\gamma_5\Phi).$$

$$\begin{Bmatrix} f(x) \\ \Delta f(x) \\ \Delta_T f(x) \end{Bmatrix} = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \begin{Bmatrix} A_1(k^2, k\cdot P) \\ A_2(k^2, k\cdot P) \\ A_3(k^2, k\cdot P) \end{Bmatrix} \delta\left(x - \frac{k^+}{P^+}\right) = \begin{cases} \int \frac{\mathrm{d}\xi^-}{4\pi}\, \mathrm{e}^{\mathrm{i}xP^+\xi^-} \langle PS|\bar{\psi}(0)\gamma^+\psi(0,\xi^-,0_\perp)|PS\rangle \\ \int \frac{\mathrm{d}\xi^-}{4\pi}\, \mathrm{e}^{\mathrm{i}xP^+\xi^-} \langle PS|\bar{\psi}(0)\gamma^+\gamma_5\psi(0,\xi^-,0_\perp)|PS\rangle \\ \int \frac{\mathrm{d}\xi^-}{4\pi}\, \mathrm{e}^{\mathrm{i}xP^+\xi^-} \langle PS|\bar{\psi}(0)\gamma^+\gamma^1\gamma_5\psi(0,\xi^-,0_\perp)|PS\rangle \end{cases}$$

Parton distribution functions

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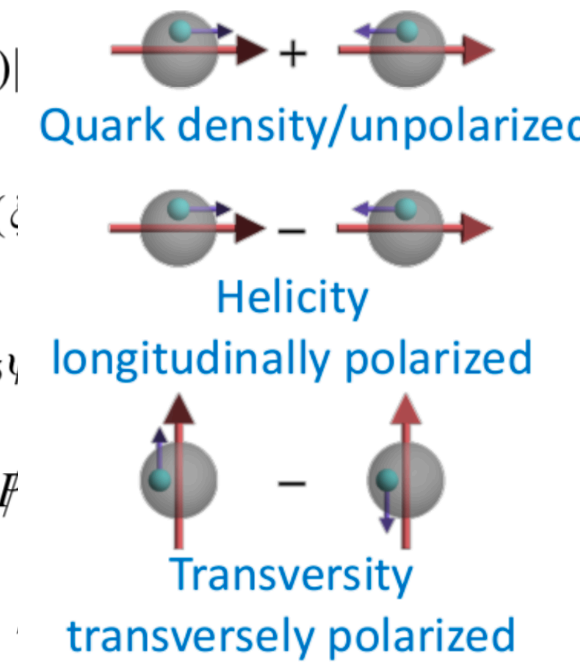


- and

$$\Phi(k,P,S) = \tfrac{1}{2}\{A_1 \not{P} + A_2\lambda_N\gamma_5\not{P}$$

$$A_1 = \frac{1}{2P^+}\mathrm{Tr}(\gamma^+\Phi), \qquad \lambda_N A_2 = \frac{1}{2P^+}\mathrm{Tr}(\gamma^+\gamma_5\Phi), \qquad \mathrm{Tr}(\gamma^+\gamma^i\gamma_5\Phi).$$

$$\left\{\begin{matrix} f(x) \\ \Delta f(x) \\ \Delta_T f(x) \end{matrix}\right\} = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\left\{\begin{matrix} A_1(k^2, k\cdot P) \\ A_2(k^2, k\cdot P) \\ A_3(k^2, k\cdot P)\end{matrix}\right\}\delta\left(x - \frac{k^+}{P^+}\right) = \left\{\begin{matrix} \int \frac{\mathrm{d}\xi^-}{4\pi}\,\mathrm{e}^{\mathrm{i}xP^+\xi^-}\langle PS|\bar{\psi}(0)\gamma^+\psi(0,\xi^-,0_\perp)|PS\rangle \\ \int \frac{\mathrm{d}\xi^-}{4\pi}\,\mathrm{e}^{\mathrm{i}xP^+\xi^-}\langle PS|\bar{\psi}(0)\gamma^+\gamma_5\psi(0,\xi^-,0_\perp)|PS\rangle \\ \int \frac{\mathrm{d}\xi^-}{4\pi}\,\mathrm{e}^{\mathrm{i}xP^+\xi^-}\langle PS|\bar{\psi}(0)\gamma^+\gamma^1\gamma_5\psi(0,\xi^-,0_\perp)|PS\rangle \end{matrix}\right.$$

Parton distribution functions

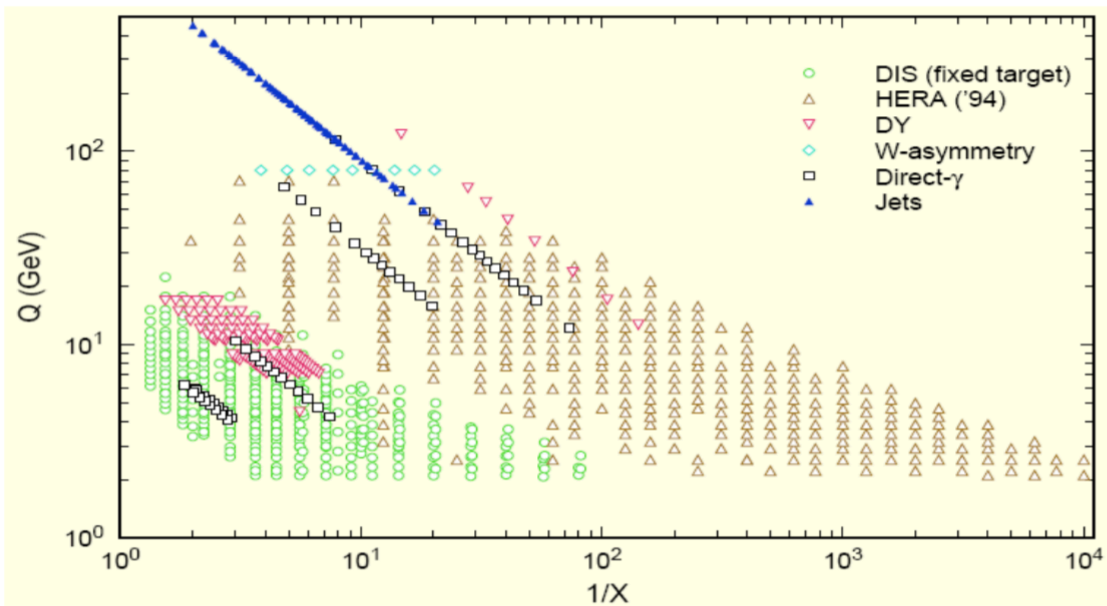
- Global determination of PDFs from experimental data

Input DPFs at Q_0

$\varphi_{f/h}\left(x,\{a_j\}\right)$

Vary $\{a_j\}$

Minimize Chi²



DGLAP

$\varphi_{f/h}(x)$ **at $Q>Q_0$**

QCD calculation

Comparison with Data at various x and Q

Procedure: Iterate to find the best set of $\{a_j\}$ for the input DPFs

Parton distribution functions

- Theory prediction from lattice QCD
- Example:

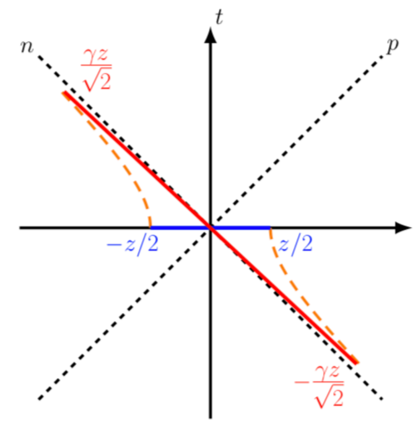
$$q(x) = \int \frac{d\lambda}{4\pi} e^{ix\lambda} \langle P | \bar{\psi}(0) n \cdot \gamma L(0,\lambda n) \psi(\lambda n) | P \rangle \qquad (n^2 = 0)$$

- When reinterpreted in Feynman's parton picture

$$q(x) = \int \frac{d\lambda}{2\pi} e^{ix\lambda} \langle P = \infty | \psi^\dagger(z) \psi(0) | P = \infty \rangle, \qquad | P = \infty \rangle = U(\Lambda_\infty) | P = 0 \rangle$$

- The boost operator can be applied either to the static operator or to the external state, projecting out the same physics
- In practice, parton physics can be approximated by static correlations at large Lorentz boost (Large-Momentum Effective Theory)

Ji, PRL 13' & SCPMA 14', Ji, Liu, Liu, JHZ, Zhao, RMP 21'

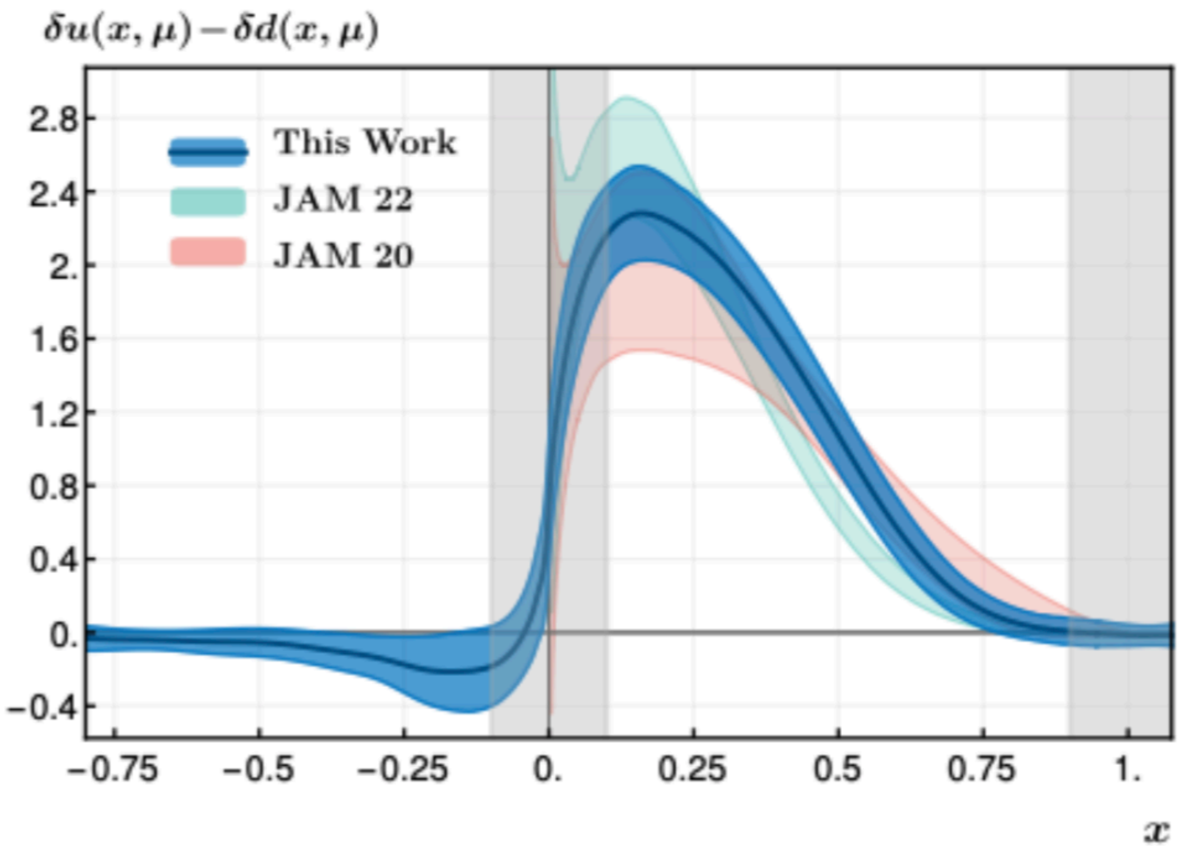


$$\tilde{q}(y, P^z) = N \int \frac{dz}{4\pi} e^{-iyzP^z} \langle P | \bar{\psi}(0) \gamma^0 L(0,z) \psi(z) | P \rangle$$

$$\tilde{q}(y, P^z) = C(y/x, \mu/xP^z) \otimes q(x, \mu) + \mathcal{O}(\Lambda_{QCD}^2/(yP^z)^2, \Lambda_{QCD}^2/((1-y)P^z)^2)$$

Parton distribution functions

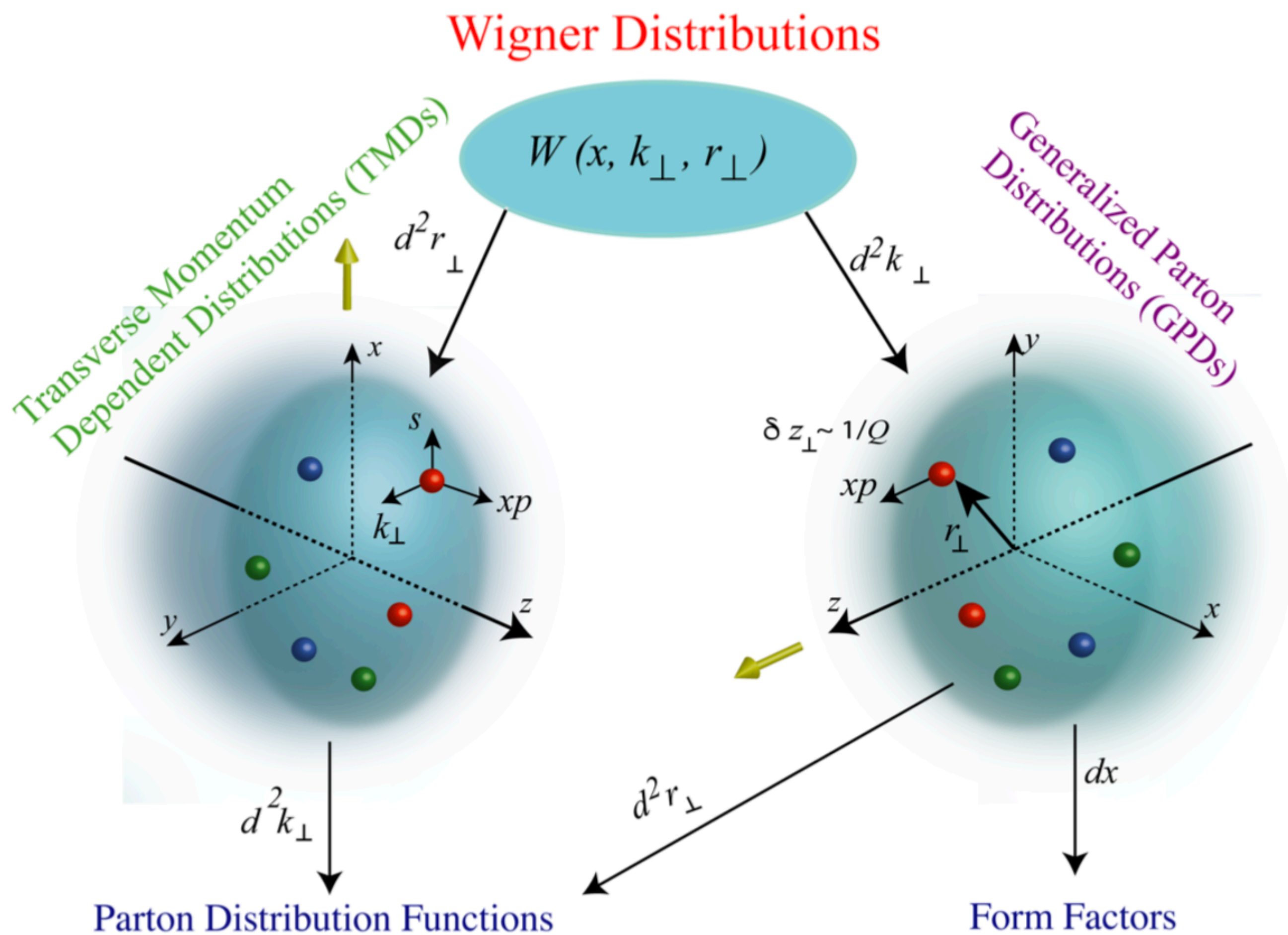
- Theory prediction from lattice QCD
- Example: nucleon isovector (u-d) quark transversity PDF



Yao et al (LPC), 22'

Generalizations: GPDs and TMDs

- PDFs can be generalized to include more kinematic dependence. The generalized quantities play an important role in describing three-dim. structure of nucleons



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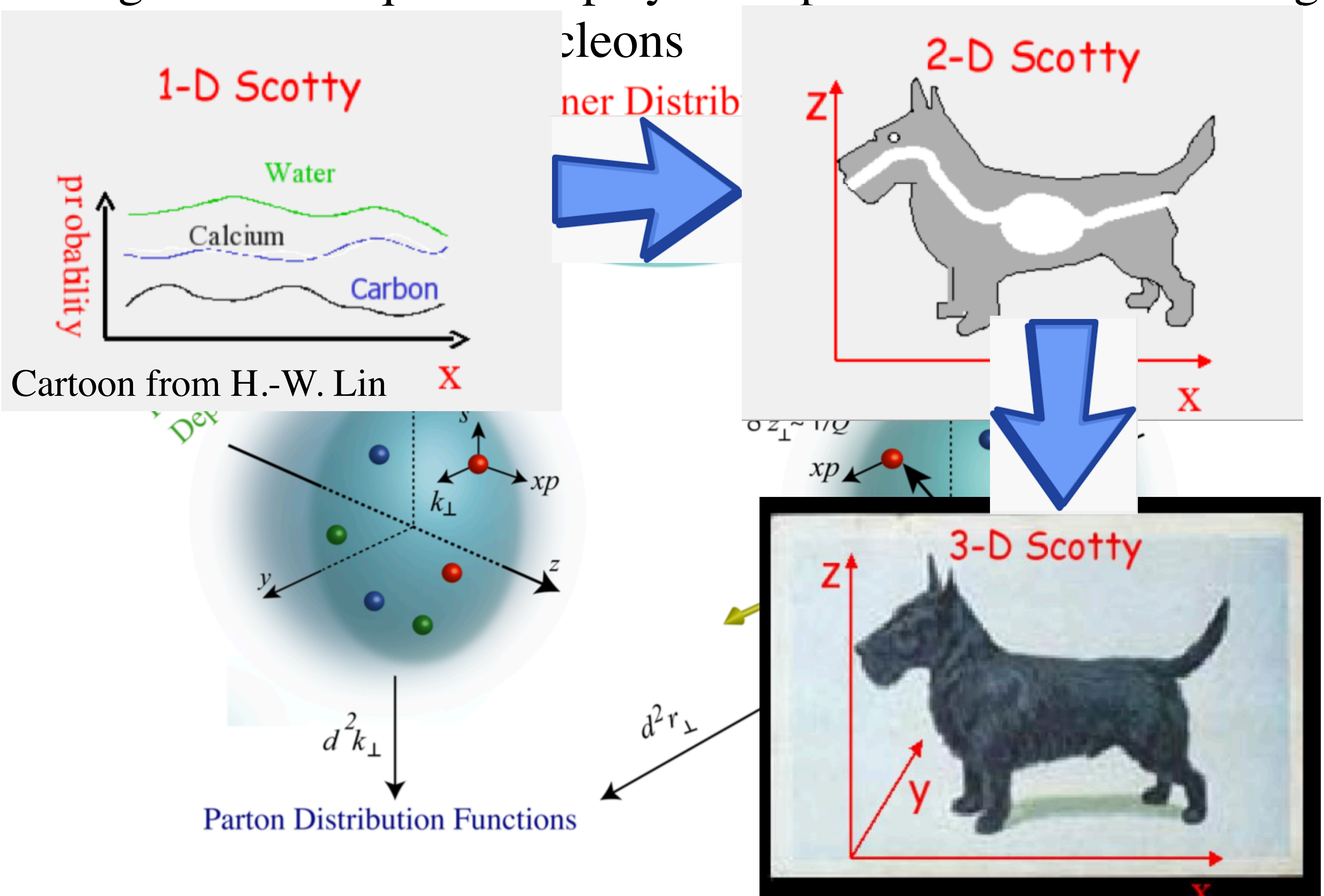


Cartoon from H.-W. Lin

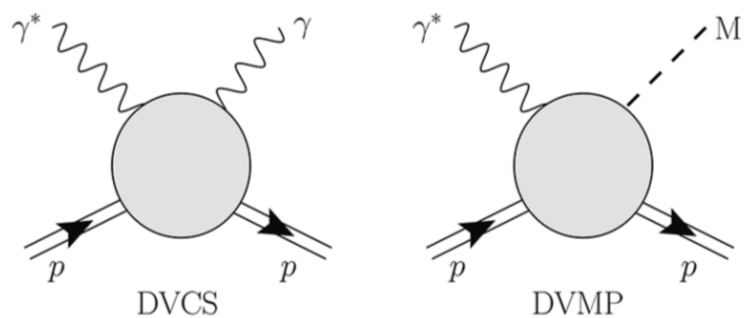
Generalizations: GPDs and TMDs

- The GPDs are given by non-forward matrix elements of nonlocal parton correlators, e.g.

$$F(x,\xi,t) = \frac{1}{2\bar{P}^+}\int \frac{d\lambda}{2\pi} e^{-ix\lambda}\langle P'|O_{\gamma^+}(\lambda n)|P\rangle = \frac{1}{2\bar{P}^+}\bar{u}(P')\left[H(x,\xi,t)\gamma^+ + E(x,\xi,t)\frac{i\sigma^{+\mu}\Delta_\mu}{2M}\right]u(P)$$

$$O_{\gamma^+}(\lambda n) = \bar{\psi}(\frac{\lambda n}{2})\gamma^+ W(\frac{\lambda n}{2}, -\frac{\lambda n}{2})\psi(-\frac{\lambda n}{2}), \quad \bar{P} = \frac{P'+P}{2}, \quad \Delta = P'-P, \quad t = \Delta^2, \quad \xi = -\frac{\Delta^+}{2\bar{P}^+}$$

- Access through exclusive processes like deeply virtual Compton scattering and meson production



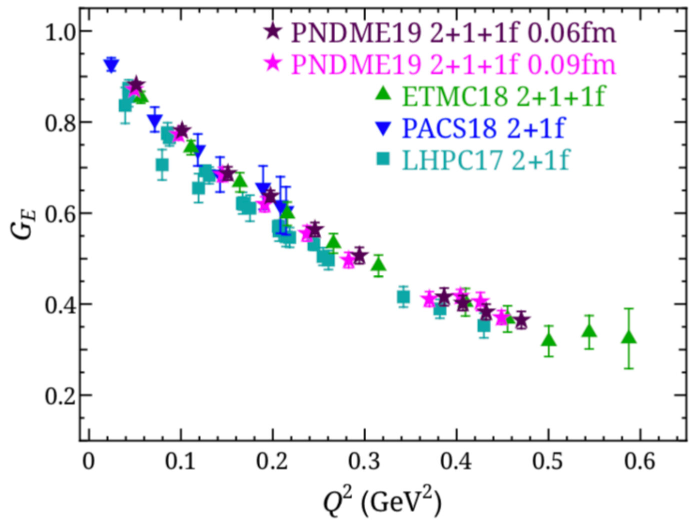
- Factorization formula

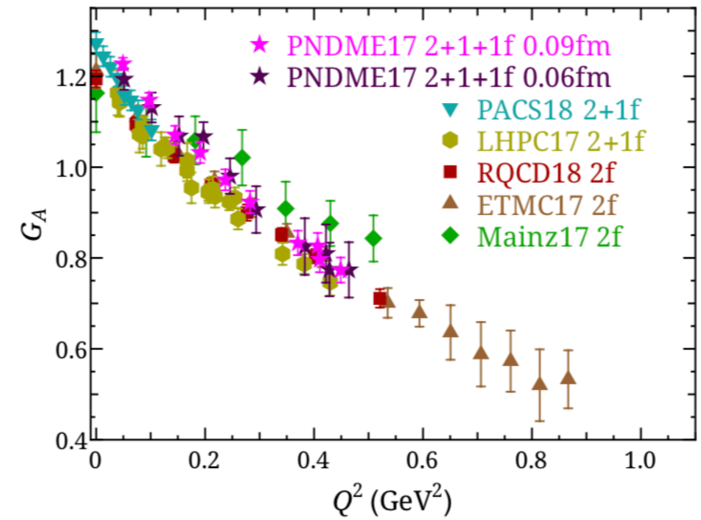
$$\mathcal{H}\left(\xi,t,Q^2\right) = \int_{-1}^{1}\frac{\mathrm{d}x}{\xi}\sum_{a=g,u,d,\ldots} C^a\left(\frac{x}{\xi},\frac{Q^2}{\mu_F^2},\alpha_S\left(\mu_F^2\right)\right) H^a\left(x,\xi,t,\mu_F^2\right)$$

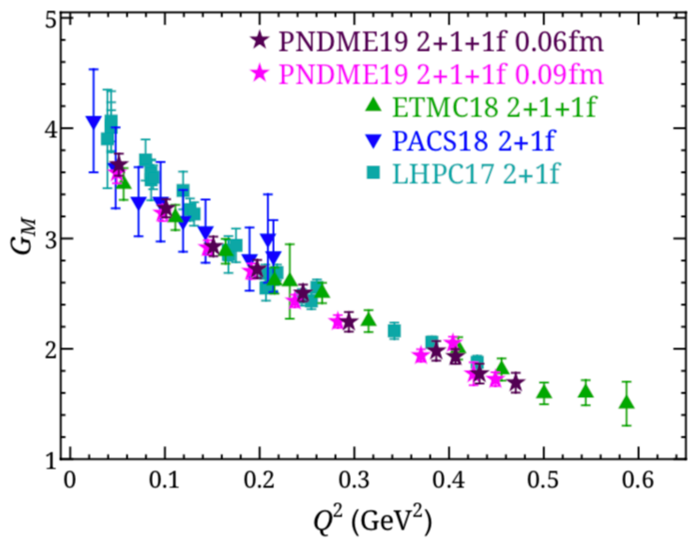
- Various models for GPD parametrization have been used for extraction from experimental data

Generalizations: GPDs and TMDs

- Form factors from nucleon GPDs $\langle x^n \rangle = \int_{-1}^{1} dx\, x^{n-1} F(x, \xi, t)$







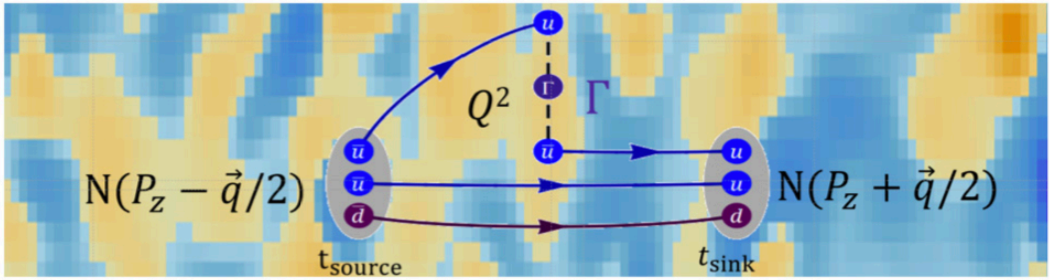
$$\langle N(p_f)|V_\mu^+(x)|N(p_i)\rangle = \bar{u}^N \left[\gamma_\mu F_1(q^2) + i\sigma_{\mu\nu}\frac{q^\nu}{2M_N}F_2(q^2)\right] u_N e^{iq\cdot x}$$

$$\langle N(p_f)|A_\mu^+(x)|N(p_i)\rangle = \bar{u}_N \left[\gamma_\mu\gamma_5 G_A(q^2) + iq_\mu\gamma_5 G_P(q^2)\right] u_N e^{iq\cdot x}$$

Constantinou, JHZ et al, Prog. Part. Nucl. Phys. 21'

Generalizations: GPDs and TMDs

- Apart from the form factors, the entire distribution can also be accessed from suitable spatial correlations on lattice



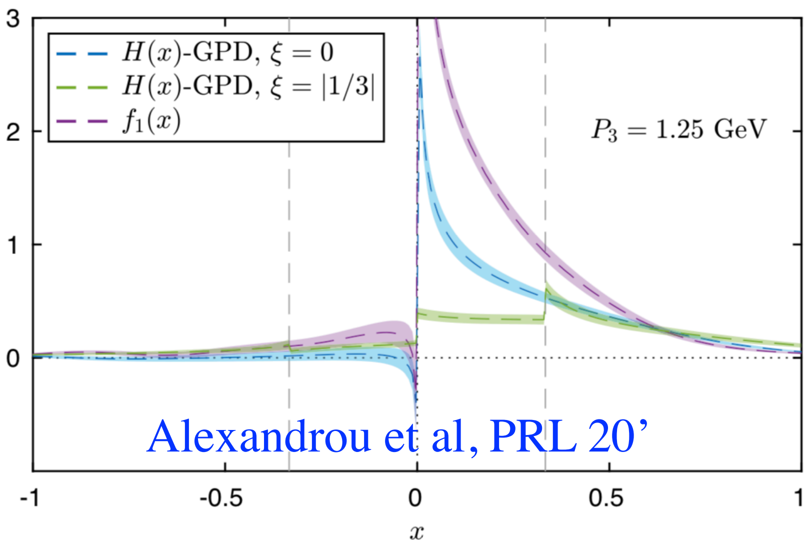
$$
\begin{aligned}
&C_\Gamma^{3pt}(\overrightarrow{p}_i, \overrightarrow{p}_f, t, t_{sep}) \\
&= |A_0|^2 \langle 0 | O_\Gamma | 0 \rangle e^{-E_0 t_{sep}} + |A_1|^2 \langle 1 | O_\Gamma | 1 \rangle e^{-E_1 t_{sep}} \\
&+ A_1 A_0^* \langle 1 | O_\Gamma | 0 \rangle e^{-E_1(t_{sep}-t)} e^{-E_0 t} + A_0 A_1^* \langle 0 | O_\Gamma | 1 \rangle e^{-E_0(t_{sep}-t)} e^{-E_1 t}
\end{aligned}
$$

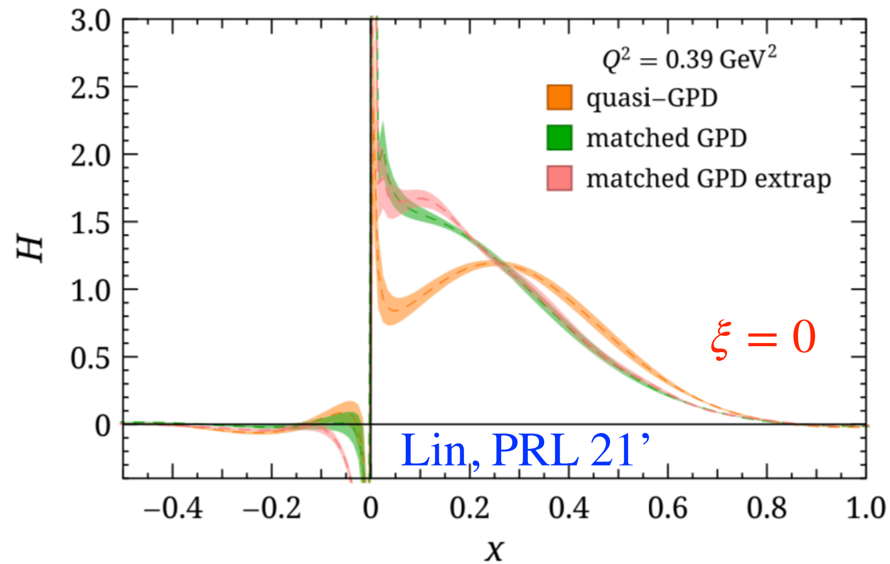
- via the factorization (after Fourier transform)

$$
\tilde{H}_{u-d}(x, \xi, t, P^z, \tilde{\mu}) = \int_{-1}^{1} \frac{dy}{|y|} C\left(\frac{x}{y}, \frac{\xi}{y}, \frac{\tilde{\mu}}{\mu}, \frac{yP^z}{\mu}\right) H_{u-d}(y, \xi, t, \mu) + h.t.
$$

Generalizations: GPDs and TMDs

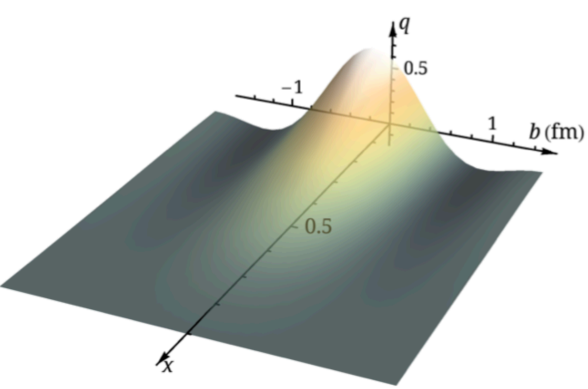
- Nucleon GPDs (unpolarized)

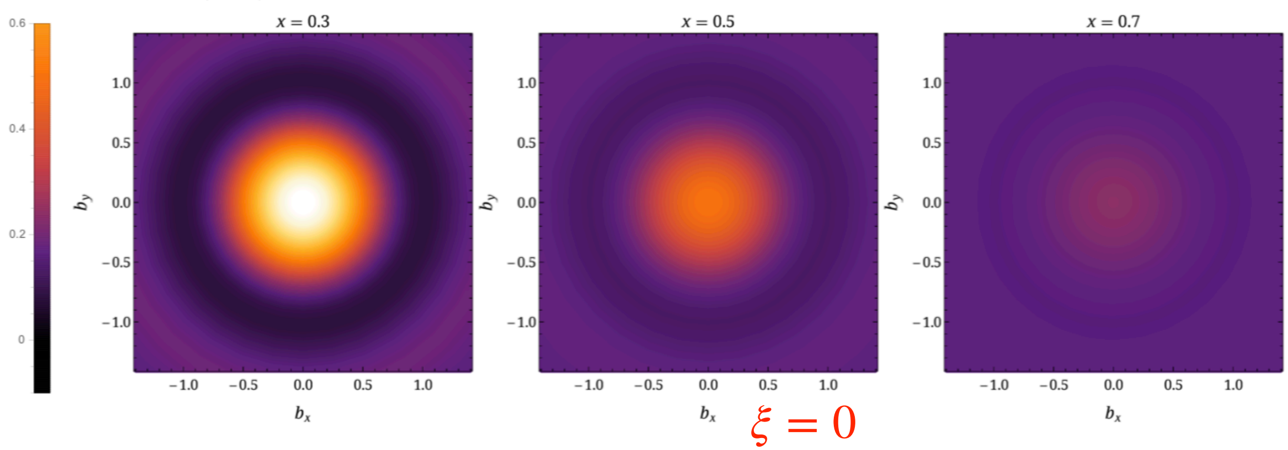




- Impact parameter distribution

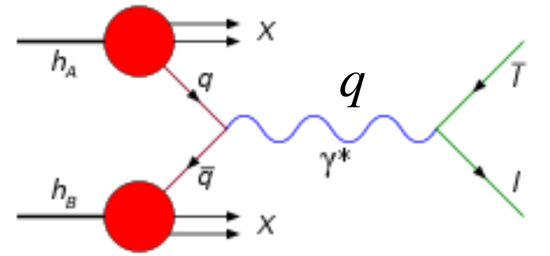
$$\mathsf{q}(x, b) = \int \frac{d\mathbf{q}}{(2\pi)^2} H(x, \xi = 0, t = -\mathbf{q}^2) e^{i\mathbf{q}\cdot\mathbf{b}}$$





Generalizations: GPDs and TMDs

- TMDs are relevant for multi-scale processes where low transverse momentum transfer is important
- Example: Drell-Yan process
- If transverse momentum $\mathbf{q_T}$ of the lepton pair is not measured



$$\frac{d\sigma}{dQ^2} = \sum_{i,j} \int_0^1 d\xi_a d\xi_b \, f_{i/P_a}(\xi_a) f_{j/P_b}(\xi_b) \frac{d\hat{\sigma}_{ij}(\xi_a, \xi_b)}{dQ^2} \times \left[1 + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}}{Q}\right)\right] \qquad Q = \sqrt{q^2}$$

- If $\mathbf{q_T}$ is measured but $|\mathbf{q}_T| \sim Q \gg \Lambda_{\rm QCD}$

$$q_T \sim Q \gg \Lambda_{\rm QCD}:$$
$$\frac{d\sigma}{dQ^2 d^2\mathbf{q_T}} = \sum_{i,j} \int_0^1 d\xi_a d\xi_b \, f_{i/P_a}(\xi_a) f_{j/P_b}(\xi_b) \frac{d\hat{\sigma}_{ij}(\xi_a, \xi_b)}{dQ^2 d^2\mathbf{q_T}} \times \left[1 + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}}{Q}, \frac{\Lambda_{\rm QCD}}{q_T}\right)\right]$$

- If $\mathbf{q_T}$ is measured but $|\mathbf{q}_T| \ll Q$

$$q_T \ll Q:$$
$$\frac{d\sigma}{dQ^2 d^2\mathbf{q_T}} = \sum_{i,j} H_{ij}(Q) \int_0^1 d\xi_a d\xi_b \int d^2\mathbf{b_T} \, e^{i\mathbf{b_T}\cdot\mathbf{q_T}} \times f_{i/P}(\xi_a, \mathbf{b_T}) f_{j/P}(\xi_b, \mathbf{b_T}) \times \left[1 + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}}{Q}, \frac{q_T}{Q}\right)\right]$$

Generalizations: GPDs and TMDs

- We need to take into account the transverse momentum of quarks

$$k^\mu \approx xP^\mu + k_\perp^\mu,\ S^\mu \approx \lambda_N \frac{P^\mu}{M} + S_\perp^\mu$$

- To leading-power accuracy, we have

$$\mathscr{V}^\mu = A_1 P^\mu,$$

$$\mathscr{A}^\mu = \lambda_N A_2 P^\mu + \frac{1}{M}\tilde{A}_1 \boldsymbol{k}_\perp \cdot \boldsymbol{S}_\perp P^\mu,$$

$$\mathscr{T}^{\mu\nu} = A_3 P^{[\mu} S_\perp^{\nu]} + \frac{\lambda_N}{M}\tilde{A}_2 P^{[\mu} k_\perp^{\nu]} + \frac{1}{M^2}\tilde{A}_3 \boldsymbol{k}_\perp \cdot \boldsymbol{S}_\perp P^{[\mu} k_\perp^{\nu]},$$

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- To leading-power accuracy, we have if time-reversal is relaxed

$$\mathscr{V}^\mu = A_1 P^\mu, + \frac{1}{M} A_1' \epsilon^{\mu\nu\rho\sigma} P_\nu k_{\perp\rho} S_{\perp\sigma}$$

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- And

$$\Phi(k,P,S) = \frac{1}{2} \Big\{ A_1 \not{P} + A_2 \lambda_N \gamma_5 \not{P} + A_3 \not{P} \gamma_5 \not{S}_\perp + \frac{1}{M} \tilde{A}_1 \boldsymbol{k}_\perp \cdot \boldsymbol{S}_\perp \gamma_5 \not{P} + \frac{1}{M} A_1' \epsilon^{\mu\nu\rho\sigma} \gamma_\mu P_\nu k_{\perp\rho} S_{\perp\sigma}$$
$$+ \frac{i}{2M} A_2' \epsilon^{\mu\nu\rho\sigma} P_\rho k_{\perp\sigma} \sigma_{\mu\nu} \gamma_5 + \tilde{A}_2 \frac{\lambda_N}{M} \not{P} \gamma_5 \not{k}_\perp + \frac{1}{M^2} \tilde{A}_3 \boldsymbol{k}_\perp \cdot \boldsymbol{S}_\perp \not{P} \gamma_5 \not{k}_\perp \Big\}.$$

- Leading-power projection is again given by

$$\frac{1}{2P^+} \mathrm{Tr}(\gamma^+ \Phi), \qquad \frac{1}{2P^+} \mathrm{Tr}(\gamma^+ \gamma_5 \Phi), \qquad \frac{1}{2P^+} \mathrm{Tr}(i\sigma^{i+} \gamma_5 \Phi)$$

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$$\left. + \frac{i}{2M} A_2' \epsilon^{\mu\nu\rho\sigma} P_\rho k_{\perp\sigma} \sigma_{\mu\nu} \gamma_5 + \tilde{A}_2 \frac{\lambda_N}{M} \not{P} \gamma_5 \not{k}_\perp + \frac{1}{M^2} \tilde{A}_3 \boldsymbol{k}_\perp \cdot \boldsymbol{S}_\perp \not{P} \gamma_5 \not{k}_\perp \right\}.$$

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$$\left. + \frac{i}{2M} A_2' \epsilon^{\mu\nu\rho\sigma} P_\rho k_{\perp\sigma} \sigma_{\mu\nu} \gamma_5 + \tilde{A}_2 \frac{\lambda_N}{M} \not{P} \gamma_5 \not{k}_\perp + \frac{1}{M^2} \tilde{A}_3 \boldsymbol{k}_\perp \cdot \boldsymbol{S}_\perp \not{P} \gamma_5 \not{k}_\perp \right\}.$$

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$$\mathscr{A}^\mu = \lambda_N A_2 P^\mu + \frac{1}{M} \tilde{A}_1 \boldsymbol{k}_\perp \cdot \boldsymbol{S}_\perp P^\mu,$$

$$\mathscr{T}^{\mu\nu} = A_3 P^{[\mu} S_\perp^{\nu]} + \frac{\lambda_N}{M} \tilde{A}_2 P^{[\mu} k_\perp^{\nu]} + \frac{1}{M^2} \tilde{A}_3 \boldsymbol{k}_\perp \cdot \boldsymbol{S}_\perp P^{[\mu} k_\perp^{\nu]}, + \frac{1}{M} A_2' \epsilon^{\mu\nu\rho\sigma} P_\rho k_{\perp\sigma}$$

- And

$$\Phi(k,P,S) = \frac{1}{2} \Big\{ A_1 \not{P} + A_2 \lambda_N \gamma_5 \not{P} + A_3 \not{P} \gamma_5 \not{S}_\perp + \frac{1}{M} \tilde{A}_1 \boldsymbol{k}_\perp \cdot \boldsymbol{S}_\perp \gamma_5 \not{P} + \frac{1}{M} A_1' \epsilon^{\mu\nu\rho\sigma} \gamma_\mu P_\nu k_{\perp\rho} S_{\perp\sigma}$$
$$+ \frac{i}{2M} A_2' \epsilon^{\mu\nu\rho\sigma} P_\rho k_{\perp\sigma} \sigma_{\mu\nu} \gamma_5 + \tilde{A}_2 \frac{\lambda_N}{M} \not{P} \gamma_5 \not{k}_\perp + \frac{1}{M^2} \tilde{A}_3 \boldsymbol{k}_\perp \cdot \boldsymbol{S}_\perp \not{P} \gamma_5 \not{k}_\perp \Big\}.$$

- Leading-power projection is again given by

$$\frac{1}{2P^+} \mathrm{Tr}(\gamma^+ \Phi), \qquad \frac{1}{2P^+} \mathrm{Tr}(\gamma^+ \gamma_5 \Phi), \qquad \frac{1}{2P^+} \mathrm{Tr}(i\sigma^{i+} \gamma_5 \Phi)$$

Generalizations: GPDs and TMDs

- These projections define the eight leading-twist quark TMDPDFs
- Introduce

$$\Phi^{[\Gamma]} \equiv \frac{1}{2}\int \frac{\mathrm{d}k^+\,\mathrm{d}k^-}{(2\pi)^4}\,\mathrm{Tr}(\Gamma\Phi)\delta(k^+ - xP^+)$$

$$= \int \frac{\mathrm{d}\xi^-\,\mathrm{d}^2\boldsymbol{\xi}_\perp}{2(2\pi)^3}\,\mathrm{e}^{\mathrm{i}(xP^+\xi^- - \boldsymbol{k}_\perp\cdot\boldsymbol{\xi}_\perp)}\langle PS|\bar{\psi}(0)\Gamma\psi(0,\xi^-,\boldsymbol{\xi}_\perp)|PS\rangle$$

- Then

$$\Phi^{[\gamma^+]} = f_1(x,\mathbf{k}_\perp^2) - \frac{\epsilon_\perp^{ij}k_{\perp i}S_{\perp j}}{M}f_{1T}^\perp(x,\mathbf{k}_\perp^2)$$

$$\Phi^{[\gamma^+\gamma_5]} = \lambda_N g_{1L}(x,\mathbf{k}_\perp^2) - \frac{\mathbf{k}_\perp\cdot\mathbf{S}_\perp}{M}g_{1T}^\perp(x,\mathbf{k}_\perp^2)$$

$$\Phi^{[i\sigma^{i+}\gamma_5]} = S_\perp^i h_1(x,\mathbf{k}_\perp^2) + \frac{\lambda_N}{M}k_\perp^i h_{1L}^\perp(x,\mathbf{k}_\perp^2) + \frac{1}{M^2}(\frac{1}{2}g_\perp^{ij}\mathbf{k}_\perp^2 - k_\perp^i k_\perp^j)S_{\perp j}h_{1T}^\perp(x,\mathbf{k}_\perp^2) - \frac{\epsilon_\perp^{ij}k_{\perp j}}{M}h_1^\perp(x,\mathbf{k}_\perp^2)$$

- The leading-twist TMDPDFs can be interpreted as number densities
- When FT to coordinate space, the correlations exhibit certain symmetries

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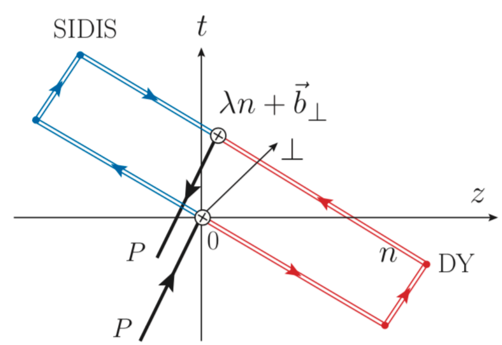
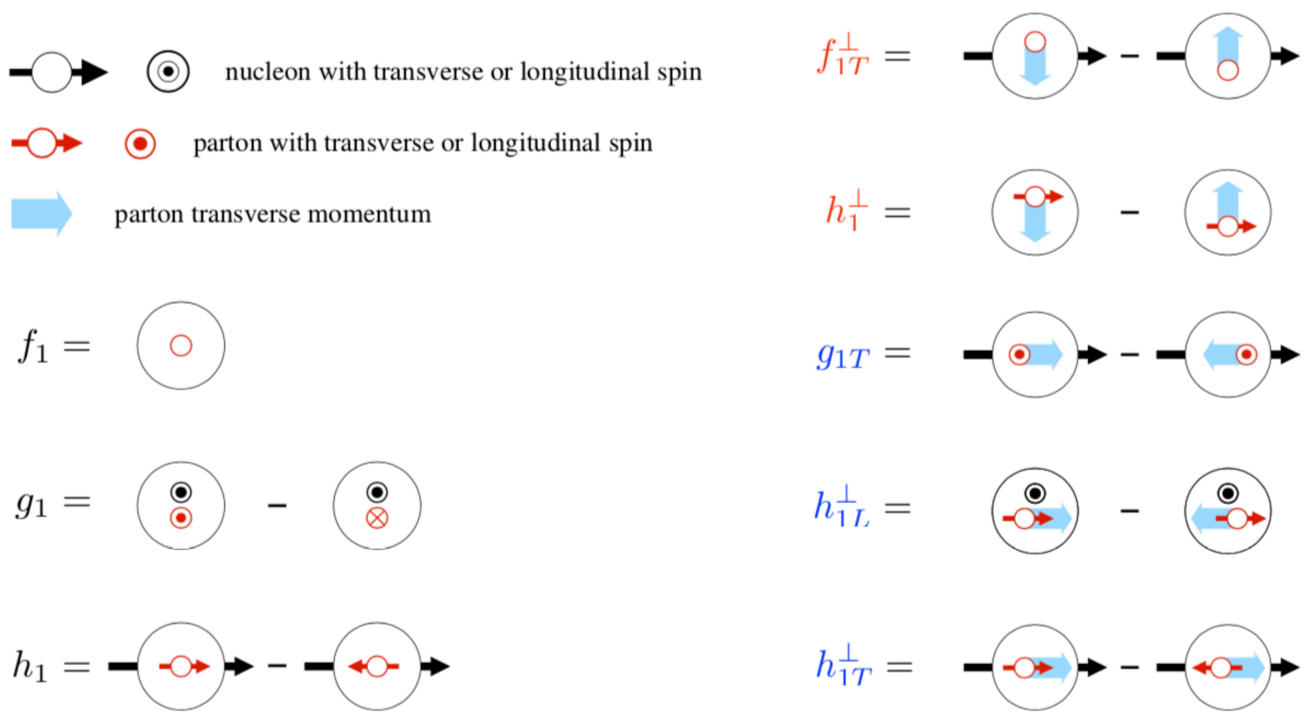
- Again, gauge links are needed to ensure gauge invariance. Now they are staple-shaped



Generalizations: GPDs and TMDs

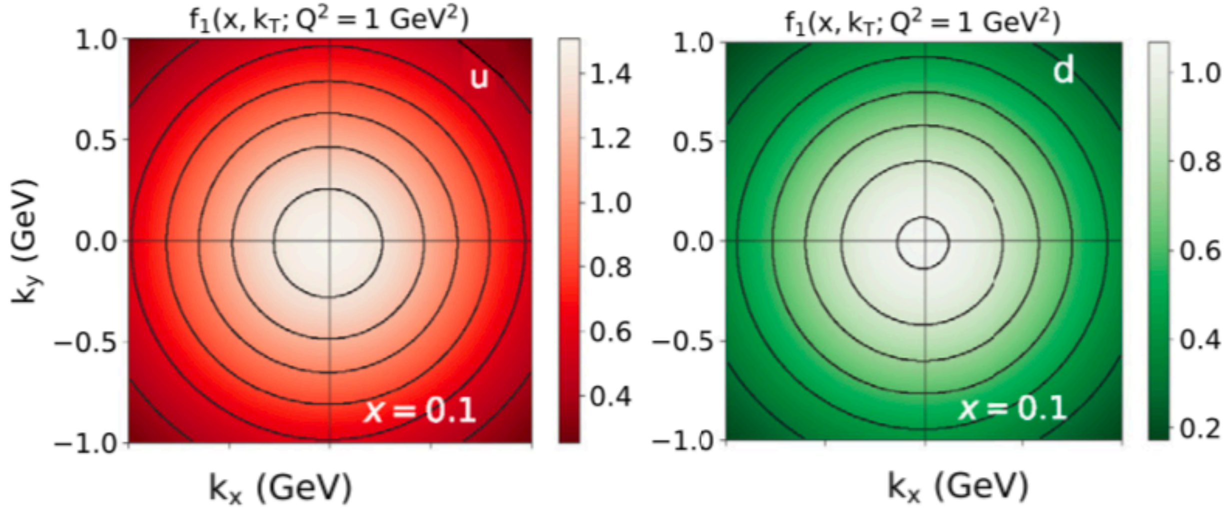
1) f_1 : unpol. TMDPDF
2) g_{1L} : helicity TMDPDF
3) h_1 : transversity TMDPDF
4) $f_{1T}^{\perp}$: Sivers function (T-odd)
5) $h_1^{\perp}$: Boer-Mulders function (T-odd)
6) $g_{1T}^{\perp}$: worm-gear T/transversal helicity TMDPDF
7) $h_{1L}^{\perp}$: worm-gear L/longitudinal transversity TMDPDF
8) $h_{1T}^{\perp}$: pretzelosity TMDPDF

nucleon pol. \ quark pol.	U	L	T	
U	f_1		$h_1^{\perp}$	
L		g_1	$h_{1L}^{\perp}$	
T	$f_{1T}^{\perp}$	g_{1T}	h_1	$h_{1T}^{\perp}$



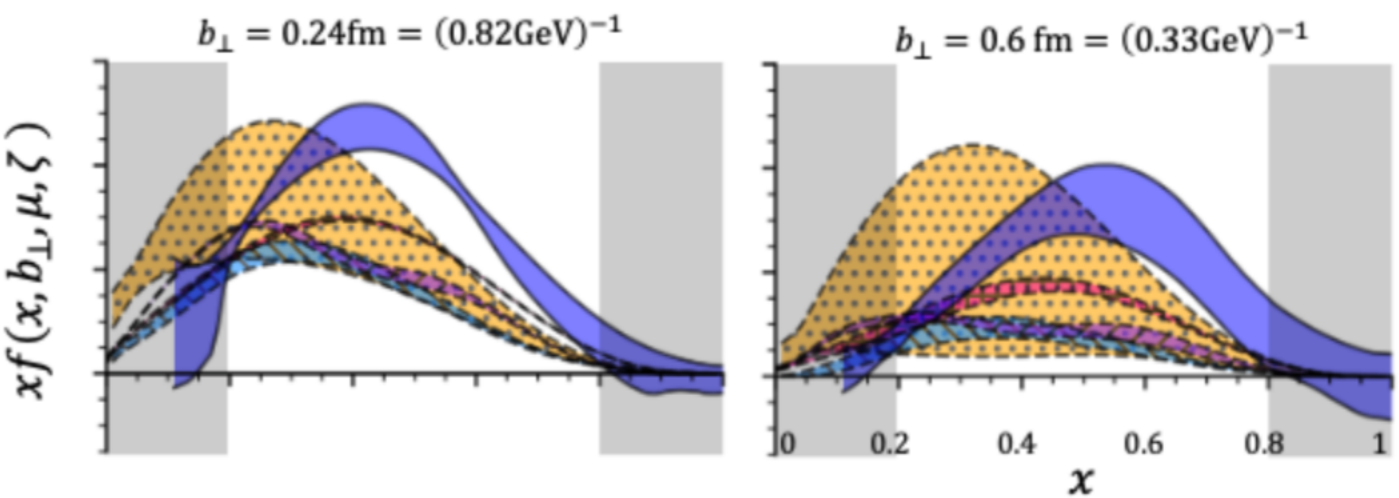
Generalizations: GPDs and TMDs

- Global analyses also exist for TMDs



- Also lattice calculations

Constantinou, JHZ et al, Prog. Part. Nucl. Phys. 21'



He et al, LPC 22'

Multiparton distributions

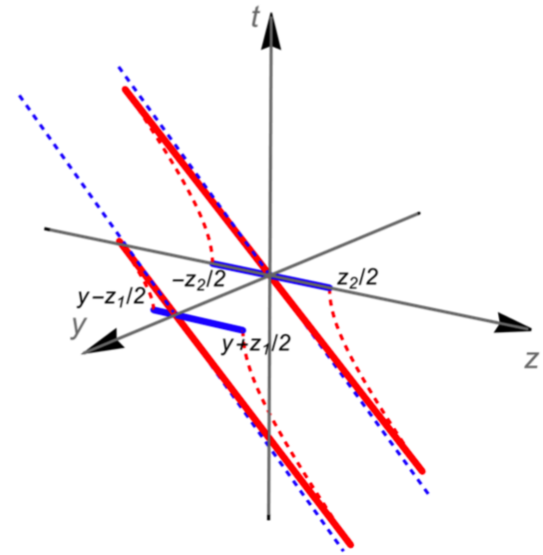
- Relevant for multiparton scattering processes
- Related to joint probability of finding two or more partons carrying momentum fractions x_i at given relative transverse separation
- Example: double parton distributions

$$f_{q_1 q_2}(x_1, x_2, y^2) = \tag{1}$$

$$2P^+ \int dy^- \int \frac{dz_1^-}{2\pi} \frac{dz_2^-}{2\pi} e^{i(x_1 z_1^- + x_2 z_2^-)P^+} h_0(y, z_1, z_2, P)$$

$$h_0(y, z_1, z_2, P) = \langle P | O_{q_1}(y, z_1) O_{q_2}(0, z_2) | P \rangle,$$

$$O_q(y, z) = \bar{\psi}_q\left(y - \frac{z}{2}\right) \frac{\gamma^+}{2} W\left(y - \frac{z}{2}; y + \frac{z}{2}\right) \psi_q\left(y + \frac{z}{2}\right),$$



JHZ, 23'

- Factorization

$$\tilde{f}(x_i, \mu_i, y^2) = C_1(x_1, x_1', \mu_1^2/(x_1' P^z)^2) \otimes C_2(x_2/x_2', \mu_2^2/x_2' P^z)^2) \otimes f(x_i', \mu_i, y^2) + \ldots$$

Summary

- Understanding the partonic structure of hadrons is an important goal of hadron physics, and is also relevant to collider phenomenology

- Lattice QCD can now be used to access dynamical properties of hadrons, and plays an important complementary role to phenomenological determinations of partonic observables
  - Form factors
  - PDFs, GPDs, TMDs…
  - Multiparton distributions

- Both analytical and numerical inputs are needed to realize such calculations

- A lot more to be explored…