The (modified) Koide formula from seesaw-type models and Yukawaon models

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The Koide formula

A relation of charged lepton mass ratios

- The Koide formula: (Koide '82, '83)

$$K = \frac{m_e + m_\mu + m_\tau}{\left(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}\right)^2} = \frac{2}{3}.$$

  - Predicting $m_\tau = 1777\text{MeV}$ from m_e and m_μ data in 1980's.
  - $m_\tau = 1783^{+3}_{-4}\text{MeV}$, DELCO '78 ($1783.5 \pm 4.2\text{MeV}$ in PDG '80) until $m_\tau = 1776.9^{+0.4}_{-0.5} \pm 0.2\text{MeV}$, BES '92.
- PDG '22 data of charged lepton masses:
  - $m_e = 0.51099895000 \pm 0.00000000015\text{MeV}$.
  - $m_\mu = 105.6583755 \pm 0.0000023\text{MeV}$.
  - $m_\tau = 1776.86 \pm 0.12\text{MeV}$ ($1776.91 \pm 0.12^{+0.10}_{-0.13}\text{MeV}$, BES3 '14).
- The Koide's character from PDG '22 (10^{-5} precision and 1-σ):

$$K = 0.6666610 \pm 0.0000068 = \frac{2}{3} \times (0.999991 \pm 0.000011).$$

Developments

Geometric visualization

- Consider $\vec{M} = \left(\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau}\right)$, $\vec{I} = (1,1,1)$, then

$$K^{-1} = \left(\vec{I} \cdot \vec{M}/\|\vec{M}\|\right)^2 = 3\cos^2\theta(\vec{I}, \vec{M}).$$

- $K = 2/3$ sets the angle $\theta(\vec{I}, \vec{M}) = \pi/4$. (Foot '94)

Radiative corrections to running masses

- QED correction shifts $K_{\rm run}$ by 10^{-3}. (Li&Ma '06, Xing&Zhang '06)
- Gauging the flavor symmetry introduces another correction which may cancel the QED correction to $K_{\rm run}$. (Sumino '08)

Analogy in quark and neutrino sectors

- Empirical formulas for masses and mixing are conjectured.
- Not as convincing as in the charged lepton sector.

Explanation

Coincidence?

- Around $K = 2/3$ is not statistically favored from random distribution of masses or Yukawa couplings.
- $K_{\text{run}} \neq 2/3$ considering radiative corrections to running masses, the known cancellation need some tuning.
- Use pole masses, running masses or on-shell masses?

Sign of new physics?

- Notice the square roots in the Koide formula, try to build a model giving the charged lepton mass matrix $M \propto \Phi\Phi$, and then $K = \text{Tr}(\Phi\Phi)/(\text{Tr}\,\Phi)^2 \equiv [\Phi\Phi]/[\Phi]^2$.
- Promote Φ to be a Hermitian nonet scalar in $\mathbf{3} \otimes \mathbf{3}^* = \mathbf{8} \oplus \mathbf{1}$ of the U(3) flavor symmetry, try to build a model setting $\langle\Phi\rangle$ ($\langle\Phi\rangle \equiv \Phi$ if there is no ambiguity) and then K.
- Try to naturally explain $K = 2/3$ from hidden symmetries.
- Candidates: the see-saw type model and the Yukawaon model.

The seesaw-type model

Seesaw to $M \propto \Phi\Phi$

- Introduce flavor nonet and singlet scalars: Φ^i_j, S and new heavy fermions: $L^i_L = (N^i_L, E^i_L)$, E^i_R. (Koide '90)
- The dimension-five effective operators:

$$\mathcal{L}^{(5)} = -\frac{y_0}{\Lambda}\left(\bar{l}_{Li}\Phi^i_j H E^j_R + \bar{L}_{Li}\Phi^i_j H e^j_R + \bar{L}_{Li} S H E^i_R\right) + \text{h.c.}.$$

- The seesaw-type mass terms from $\langle H\rangle = \left(0, v/\sqrt{2}\right)$:

$$\begin{aligned}\mathcal{L} &= -\bar{e}_L m_L E_R - \bar{E}_L m_R e_R - \bar{E}_L M_E E_R + \text{h.c.}\\ &= -\begin{pmatrix}\bar{e}_{Li} & \bar{E}_{Lj}\end{pmatrix}\begin{pmatrix}0 & m^i_{Ll}\\ m^j_{Rk} & M_E\delta^j_l\end{pmatrix}\begin{pmatrix}e^k_R\\ E^l_R\end{pmatrix} + \text{h.c.},\end{aligned}$$

  with $m^i_{Ll} = m^i_{Rl} = \frac{y_0 v}{\sqrt{2}\Lambda}\Phi^i_j$, $M_E = \frac{y_0 v}{\sqrt{2}\Lambda}S$.
- Giving $M_E \gg \|m_L\| = \|m_R\|$, the block-diagonalized mass matrix: $M^i_{Ej} \approx M_E\delta^i_j$, $M^i_{ej} \approx m^i_{Lk} M_E^{-1} m^k_{Rj} = \frac{y_0 v}{\sqrt{2}\Lambda S}\Phi^i_k\Phi^k_j$.

The Yukawaon model

SUSY Yukawaons to $M \propto \Phi\Phi$

- Introduce two flavor nonet scalars: the Yukawaon Y^i_j replacing the Yukawa coupling coefficients and the ur-Yukawaon Φ^i_j, both are promoted to chiral superfields. (Koide '08)
- The dimension-five effective operators and the superpotential:

$$\mathcal{L}^{(5)} = -\frac{y_0}{\Lambda} \bar{l}_{Li} Y^i_j H e^j_R + \text{h.c.},$$

$$W_0 = \lambda_A [\Phi\Phi A] + \mu_A [Y A] + W(\Phi, \phi_a),$$

  with another flavor nonet A^i_j and more chiral superfields ϕ_a.
- At a SUSY vacuum, the F-term equations $\partial_A W_0 = \partial_Y W_0 = 0$ set $A^i_j = 0$ and $Y^i_j = -\frac{\lambda_A}{\mu_A} \Phi^i_k \Phi^k_j$.
- $\langle H \rangle = \left(0, v/\sqrt{2}\right)$ leads to the charged lepton mass matrix: $M^i_{ej} = \frac{y_0 v}{\sqrt{2}\Lambda} Y^i_j = -\frac{y_0 \lambda_A v}{\sqrt{2}\Lambda\mu_A} \Phi^i_k \Phi^k_j$.
- The F-term equations $\partial_\Phi W = \partial_{\phi_a} W = 0$ fix Φ.

The superpotential from symmetries

$W(\Phi, \phi_a)$ for $\langle \Phi \rangle$

- The superpotential: (Liang&Sun '20)

$$
\begin{aligned}
W &= W_1 + W_2, \\
W_1 &= \frac{1}{2} \begin{pmatrix} \phi_1' & \phi_2' \end{pmatrix} \begin{pmatrix} \mu_{11}' & \mu_{12}' \\ \mu_{12}' & \mu_{22}' \end{pmatrix} \begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} \\
&\quad + \begin{pmatrix} \phi_1' & \phi_2' \end{pmatrix} \begin{pmatrix} b_{11}' & b_{12}' \\ b_{21}' & b_{22}' \end{pmatrix} \begin{pmatrix} [\Phi_8 \Phi_8] \\ [\Phi]^2 \end{pmatrix}, \\
W_2 &= \mu_0 [\Phi\Phi] = \mu_0 [\Phi_8 \Phi_8] + \frac{1}{3} \mu_0 [\Phi]^2.
\end{aligned}
$$

- Introducing an R-symmetry, ϕ_1' and ϕ_2' has charge 1, the nonet $\Phi = \Phi_8 + \frac{1}{3}[\Phi]\mathbf{I}_{3\times 3} = \Phi_8^a t^a + [\Phi] t^0$ has charge $1/2$.
- All renormalizable terms respecting the flavor symmetry and the R-symmetry are included in W_1.
- W_2 breaks the R-symmetry with a nonzero μ_0.

Simplification

Simplifying W by a field redefinition

- Off-diagonalize the quadratic part of W_1:

$$P^{\mathsf{T}} \begin{pmatrix} \mu'_{11} & \mu'_{12} \\ \mu'_{12} & \mu'_{22} \end{pmatrix} P = \begin{pmatrix} 0 & \mu_3 \\ \mu_3 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} \frac{\mu_3 \mu_\pm}{2\Delta} & -\frac{\mu_{22}}{\mu_\pm} \\ -\frac{\mu_3 \mu_{11}}{2\Delta} & 1 \end{pmatrix},$$

  with $\Delta = \mu_{12}^2 - \mu_{11}\mu_{22}, \ \mu_\pm = \mu_{12} \pm \sqrt{\Delta}$.

- The field and coefficient redefinition:

$$\begin{pmatrix} \phi'_1 \\ \phi'_2 \end{pmatrix} = P \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad \begin{pmatrix} b'_{11} & b'_{12} \\ b'_{21} & b'_{22} \end{pmatrix} = \left(P^{\mathsf{T}}\right)^{-1} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

- W is simplified after the redefinition:

$$W = \mu_0[\Phi\Phi] + \mu_3 \phi_1 \phi_2 + \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} [\Phi\Phi] - \frac{1}{3}[\Phi]^2 \\ [\Phi]^2 \end{pmatrix}.$$

Towards the modified Koide formula

Setting Φ and K

- $\Phi \in \mathfrak{u}(3)_{\mathbb{C}} \cong \mathfrak{gl}(3, \mathbb{C}) = \mathbb{C}^{3\times 3}$, so the F-term equations for Φ: $\partial_{\Phi_8} W = \partial_{[\Phi]} W = 0$ are equivalent to $\partial_{\Phi^i_j} W = 0$.
- Assuming Φ gets a non-zero Hermitian expectation value, the F-term equations $\partial_{\Phi} W = \partial_{\phi_a} W = 0$ lead to

$$K = \frac{[\Phi\Phi]}{[\Phi]^2} = \frac{2}{3} \times \left(1 - \frac{a_{02} - 9a_4}{2a_{02} - 3a_2}\right),$$

$$\mu_0\mu_3 = \left(2a_{02}\left(K - \frac{1}{3}\right) + a_2\right)[\Phi]^2,$$

  with $a_{02} = b_{11}b_{21},\ a_2 = b_{11}b_{22} + b_{12}b_{21},\ a_4 = b_{12}b_{22}$.
- K is modified by two effective parameters a_{02}/a_2 and a_4/a_2.
- Nonzero μ_0 and μ_3 are generally needed for a nonzero Φ.
- Building a scalar potential can also set a non-zero Φ, but it is not protected by the SUSY non-renormalization theorem.

Discussion

Tuning or lack of prediction power?

- The Cauchy-Schwarz inequality and positive mass condition leads to $K \in [1/3, 1]$, corresponding to $\frac{a_{02}-9a_4}{2a_{02}-3a_2} \in [-1/2, 1/2]$.
- Adjusting parameters can fit any K in the range.
- $K = 2/3$ corresponds to $a_{02} - 9a_4 = 0$.
- In particular, $a_{02} = a_4 = 0$ corresponds to the superpotential:

$$W = \mu_0[\Phi\Phi] + \mu_3\phi_1\phi_2 + b_{11}\phi_1[\Phi_8\Phi_8] + b_{22}\phi_2[\Phi]^2,$$

  which is used in previous literature. (Koide '18)
- Currently no satisfactory reason to choose such a W.

Other pheno

- Pheno from the dimension-five operators or UV completion?
- Gauged flavor symmetry to cancel raidative corrections?