Fractal Path Integral and its Degeneration to Dimensional Regularization
分形路径积分与维数正规化

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1. Motivation
2. Background Knowledge
3. Geometry with non-integer dimensions and its application in physics
4. Fractal Path Integral
5. Summary and Prospect
6. Backup

Field quantization based on fractal paths integral

- Particles can propagate under fractal paths, such as Brownian motion.
- Path integral method sums all possible intermediate paths. Superposition principle.
- Traditional path integral method does not include paths with non-integer dimensions.
- We extend the path integral method involving non-integer fractal paths.

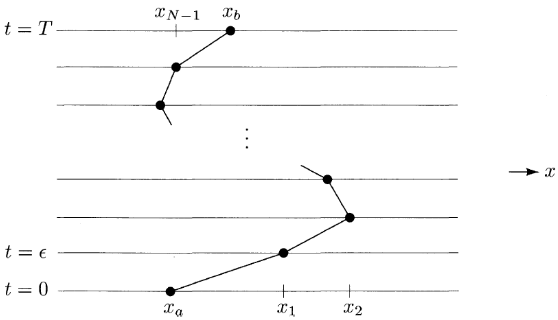

图 1: Path integral of polygonal line.

Key feature of fractal: self similarity.
Traditional differential and integral calculus is not applicable for fractal paths. Non-integer dimension fractal derivative is needed.

Application of fractal in physics

- Diffusion Equation that describe Brownian motion.
- Dimensional Regularization based on non-integer spacetime in QFT.
- QFT in non-integer spacetime and multi-non-integer spacetime.

Hausdorff Dimension and non-integer order derivative

Hausdorff dimension is defined according to the scale relation for a geometry object. Eg. for Koch curve, its Hausdorff dimension is $\log_3 4$.

For fractal geometry, its Hausdorff dimension is usually not equal to its Topological dimension. In physics, equations with non-integer order derivatives will be used in describing motions with fractal paths.

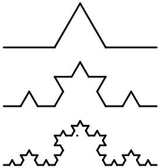

图 2: Koch curve.

Fractal calculus

Cauchy formula for the n-time repeated integral

$$(I^n f)(x) = \int_{x_0}^{x} dy_1 \int_{x_0}^{y_2} \cdots \int_{x_0}^{y_{n-1}} dy_n f(y_n) \tag{1}$$

$$= \frac{1}{(n-1)!} \int_{x_0}^{x} dx'(x-x')^{n-1} f(x')\,, \tag{2}$$

Analytic Continuation. The left fractal integral

$$(I^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x_1} \frac{dx'}{(x-x')^{1-\alpha}} \theta(x-x') f(x'), \tag{3}$$

The right fractal integral

$$(\bar{I}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x_1} \frac{dx'}{(x'-x)^{1-\alpha}} \theta(x'-x) f(x'). \tag{4}$$

Fractal calculus

The left and right Caputo derivatives

$$(\partial^\alpha f)(x) = (I^{n-\alpha}\partial^n)f(x) = \frac{1}{\Gamma(n-\alpha)}\int_{x_0}^{x_1}\frac{dx'}{(x-x')^{\alpha+1-n}}\theta(x-x')\partial^n_{x'}f(x'). \tag{5}$$

$$(\bar{\partial}^\alpha f)(x) = (\bar{I}^{n-\alpha}\partial^n)f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_{x_0}^{x_1}\frac{dx'}{(x'-x)^{\alpha+1-n}}\theta(x'-x)\partial^n_{x'}f(x'), \tag{6}$$

Liouville and Weyl fractal derivative:

$$(\partial^\alpha f)(x) = \frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{x}\frac{dx'}{(x-x')^\alpha}\partial f(x'), \tag{7}$$

$$(\bar{\partial}^\alpha f)(x) = \frac{-1}{\Gamma(1-\alpha)}\int_{x}^{+\infty}\frac{dx'}{(x'-x)^\alpha}\partial f(x'). \tag{8}$$

Correspondence between micro motions and macro dynamics

Brownian motion and equation of diffusion：

$$\frac{\partial p(x,t)}{\partial t} = \frac{\sigma}{2}\nabla^2 p(x,t) \tag{9}$$

Its solution：

$$\frac{1}{\sqrt{2\pi\sigma(t-t_0)}} \exp - \frac{(x-x_0)^2}{2\sigma(t-t_0)} \tag{10}$$

scale relation $(x-x_0)^2 \sim \sigma(t-t_0)$， $d_H = 2$， Topological dimension is 1, character of fractal.
Advection equation 对流方程：

$$\frac{\partial c}{\partial t} = -v \cdot \nabla c \tag{11}$$

$d_H = 1$， Topological dimension 1， so the micro motion for advection equation is not fractal.

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   - Action and propagator
   - Tree level and one loop level calculation.
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Action

By using non-integer derivative, different dynamics can described uniformly. We define dynamic dimension d_f。
For action of scalar fields,

$$S_0 = \int d^4x \mathcal{L}_0 = \frac{1}{2}\int d^4x[(\partial_\mu\phi)^2 - m^2\phi^2]. \tag{12}$$

Scale relation between field function and space time coordinate:
$(\triangle\phi)^2 \sim \triangle x^\mu$, $\rightarrow \mathrm{d}_f = 2$。

- We extend the action of scalar fields as：

$$S_0 = \frac{1}{2}\int d^4x \left[\beta^2 i^{2-2\alpha}\phi\partial_\mu^\alpha\bar{\partial}_\mu^\alpha\phi - m^2\phi^2\right] \tag{13}$$

where β is a scale factor with with mass dimension $m^{1-\alpha}$, $\beta i^{1-\alpha}\partial_\mu^\alpha$ is fractal momentum operator, $d_f = \frac{2\alpha}{2\alpha-1}$, and $\alpha \in (\frac{1}{2}, 1]$.

Action

- For vector fields：

$$S_0 = -\frac{1}{2}\eta^2 i^{2-2\alpha} \int d^4 x A^\mu (\partial_\lambda^\alpha \bar{\partial}_\lambda^\alpha g_{\mu\nu} - \partial_\mu^\alpha \bar{\partial}_\nu^\alpha) A^\nu \qquad (14)$$

- For spinor fields：

$$S_0 = \frac{1}{2}\int d^4 x \bar{\psi}(i^{2-\alpha}\kappa \not{\partial}^\alpha - m)\psi, \qquad (15)$$

η and κ are scale factors similar as β for scalar fields.

Equation of motion

$$
\begin{aligned}
0 = \delta S &= \int d^4x[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu^\alpha \phi)}\delta(\partial_\mu^\alpha \phi)] \\
&= \int d^4x[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \bar{\partial}_\mu^\alpha(\frac{\partial \mathcal{L}}{\partial(\partial_\mu^\alpha \phi)})\delta\phi],
\end{aligned} \quad (16)
$$

The E-L equation

$$
\frac{\partial \mathcal{L}}{\partial \phi} + \bar{\partial}_\mu^\alpha(\frac{\partial \mathcal{L}}{\partial(\partial_\mu^\alpha \phi)}) = 0, \quad (17)
$$

Scalar field

$$
\beta^2 i^{2-2\alpha}\bar{\partial}_\mu^\alpha \partial_\mu^\alpha \phi - m^2\phi = 0, \quad (18)
$$

Vector field

$$
\bar{\partial}\partial A_\nu - \partial_\nu^\alpha \bar{\partial}^{\mu\alpha} A_\mu = 0, \quad (19)
$$

Spinor field

$$
(i^{2-\alpha}\kappa \not{\partial}^\alpha - m)\psi = 0. \quad (20)
$$

Propagators

- Propagator for scalar field：

$$\int \frac{d^4k}{(2\pi)^4} \frac{-ie^{-ik\cdot(x_1-x_2)}}{\beta^2 k^\alpha(-k)^\alpha + m^2 + i\epsilon} \tag{21}$$

- Propagator for vector field：

$$\frac{ig_{\mu\nu}}{\eta^2 k^\alpha(-k)^\alpha + i\epsilon}. \tag{22}$$

- Propagator for spinor field：

$$\int \frac{d^4k}{(2\pi)^4} \frac{i(-\kappa(-\not{k})^\alpha + m)}{\kappa^2(-k)^{2\alpha} - m^2 + i\epsilon} e^{-ik\cdot(x_1-x_2)}. \tag{23}$$

P parity conservation for scalar and vector propagators, not for spinor propagator.

Tree level $e^+e^- \to \mu^+\mu^-$

- Amplitude for $e^+e^- \to \mu^+\mu^-$:

$$\frac{1}{4}\sum_s |\mathcal{M}|^2 = \frac{8e^4}{16\eta^4 E^{4\alpha}}[E^2(E-|k|\cos\theta)^2 + E^2(E+|k|\cos\theta)^2 + 2m_\mu^2 E^2]. \tag{24}$$

This result shows the dependence of the α and η in $|\mathcal{M}|^2$. When $\alpha \to 1$, $\eta = 1$, it goes back to ordinary QED result.

Self energy for the electron

Loop momentum integral：

$$I = \int d^4 k \gamma^\mu \frac{(-\not{k})^\alpha - m_1}{(-k)^{2\alpha} - m_1^2 + i\epsilon} \gamma_\mu \frac{1}{(p-k)^\alpha (k-p)^\alpha + i\epsilon} = (-1)^{-\alpha} I_1 \quad (25)$$

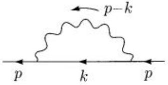

图 3: Self energy diagram for the electron

$$(k-p)^{2\alpha} = k^{2\alpha} - 2\alpha p k^{2\alpha-1} + c, \quad c = \begin{cases} 0 & \alpha \in (\frac{1}{2}, 1) \\ p^2 & \alpha = 1. \end{cases} \quad (26)$$

Calculate this integral by Feynman parameterization.

$\triangle = -x(1-x)c + x m_1^2.$

Electron self energy and dimensional regularization

After some calculation：

$$I_1 = -2(2m_1 I_2 + \not{z}^{-1} I_3). \tag{27}$$

where

$$I_2 = \frac{i^{\frac{1}{\alpha}}}{2\alpha} \int d\Omega(\xi^{-\frac{2}{\alpha}}) \triangle^{\frac{2-2\alpha}{\alpha}} \frac{\Gamma(\frac{2\alpha-2}{\alpha})\Gamma(\frac{2}{\alpha})}{\Gamma(2)}, \tag{28}$$

$$I_3 = \frac{1}{2\alpha} \int d\Omega(\cos^\alpha \theta \xi^{-\frac{4+\alpha}{2\alpha}}) \triangle^{\frac{4-3\alpha}{2\alpha}} \frac{\Gamma(\frac{3}{2} - \frac{2}{\alpha})\Gamma(\frac{1}{2} + \frac{2}{\alpha})}{\Gamma(2)} + b I_2, \tag{29}$$

$$\xi = \cos^{2\alpha}\theta + \sin^{2\alpha}\theta[\cos^{2\alpha}\phi + \sin^{2\alpha}\phi(\cos^{2\alpha}\omega + \sin^{2\alpha}\omega)].$$

Electron self energy and dimensional regularization

In dimensional regularization

$$I_1 = 2i\pi^{\frac{d}{2}}[(1-x)\not{p} - 2m_1]\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}(\frac{1}{\triangle})^{2-\frac{d}{2}} \tag{30}$$

- Fractal path integral $\alpha \to 1$:

$$\begin{aligned} I_1 &= -2(2m_1 I_2 + \not{z}^{-1} I_3) \\ &= 2i\pi^2[(1-x)\not{p} - 2m_1]\Gamma(0). \end{aligned} \tag{31}$$

- Dimensional regularization $d \to 4$:

$$\begin{aligned} I_1 &= 2i\pi^{\frac{d}{2}}[(1-x)\not{p} - 2m_1]\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}(\frac{1}{\triangle})^{2-\frac{d}{2}} \\ &= 2i\pi^2[(1-x)\not{p} - 2m_1]\Gamma(0). \end{aligned} \tag{32}$$

Passarino-Veltman Reduction

One loop momentum integral can be written in a uniform：

$$T^N_{\mu_1\cdots\mu_p}(p_1,\cdots,p_{N-1},m_0,\cdots,m_{N-1}) = \frac{1}{i\pi^2}\int d^4q\frac{q_{\mu_1}\cdots q_{\mu_p}}{D_0D_1\cdots D_{N-1}}, \tag{33}$$

T^N at $N > 4$ can be written in a linear combination of T^1, T^2, T^3, and T^4, which can be expanded according to Eq.(23) and calculated by Feynman parametrization. A systematic treatment of one loop diagram by fractal path integral equivalent to the dimensional regularization.

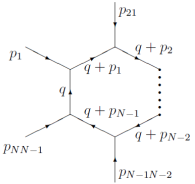

图 4: Conventions for the N-point

Summary and Prospect

- We construct a method of fractal path integral, in which particle paths can have continuous dynamical dimension. We find one possible utilization is that it can be equivalent to dimensional regularization in one loop diagram calculation.
- We show the modified equation of motion for scalar, vector and spinor fields in the framework of fractal path integral.
- Non-local gauge symmetry caused by the fractal derivative. Medium effect?

Thanks!

Linear expansion of non-integer order power function

$$
\begin{aligned}
\partial^{\alpha}(x-x_*)^{\beta} &= \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)}\int_{-\infty}^{x} dx' \frac{(x'-x_*)^{\beta-n}}{(x-x')^{\alpha+1-n}} \\
&= \frac{(-1)^{\beta-n}\Gamma(\beta+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)}\int_{-x}^{+\infty} dy \frac{(y+x_*)^{\beta-n}}{(y+x)^{\alpha+1-n}} \\
&= \frac{(-1)^{\beta-n}\Gamma(\beta+1)\Gamma(\alpha-\beta)}{\Gamma(1-n+\beta)\Gamma(n-\beta)}(x_*-x)^{\beta-\alpha} \\
&= (-1)^{\alpha}\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}\frac{\sin(\pi\beta)}{\sin[\pi(\beta-\alpha)]}(x-x_*)^{\beta-\alpha}.
\end{aligned}
\tag{34}
$$

Linear expansion of non-integer order power function

in which use the integral formula:

$$\int_a^{\infty} (x+b)^{-\nu}(x-a)^{\mu-1}dx = (a+b)^{\mu-\nu}B(\nu-\mu,\mu) \tag{35}$$

and

$$\Gamma(1-x)\Gamma(x) = \frac{\pi}{sin(\pi x)} \tag{36}$$

Beta function $B(p,q)$ is defined in the region $p>0, q>0$, we need to analytic continued it to the hole plane. Relation between *Beta* function and Γ function:

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \tag{37}$$

Using the property of the Γ in the negative half axis, *Beta* function can be analytic continued to the hole plane. The derivative Liouvil formula to order function exists for arbitrary α, β.

Linear expansion of non-integer order power function

Using the linearity of Liouville and Weyl fractal derivative operator, $(k-p)^{2\alpha}$ can be expanded as:

$$\begin{aligned}
\bar{\partial}^{2-2\alpha}(k-p)^2 &= \frac{\Gamma(3)}{\Gamma(2\alpha+1)}\frac{\sin(2\pi)}{\sin(2\alpha\pi)}(k-p)^{2\alpha} \\
&= \bar{\partial}^{2-2\alpha}k^2 - \bar{\partial}^{2-2\alpha}2kp + \bar{\partial}^{2-2\alpha}p^2 \\
&= \frac{\Gamma(3)}{\Gamma(2\alpha+1)}\frac{\sin(2\pi)}{\sin(2\alpha\pi)}k^{2\alpha} \\
&\quad - \frac{\Gamma(2)}{\Gamma(2\alpha)}\frac{\sin(\pi)}{\sin[(2\alpha-1)\pi]}2pk^{2\alpha-1} + c, \\
c &= \begin{cases} 0 & \alpha \neq 1 \\ p^2 & \alpha = 1. \end{cases}
\end{aligned} \tag{38}$$

Linear expansion of non-integer order power function

Comparing the first line and the above last equation, we obtain:

$$(k-p)^{2\alpha} = k^{2\alpha} - 2\alpha p k^{2\alpha-1} + c, \quad c = \begin{cases} 0 & \alpha \in (\frac{1}{2}, 1) \\ p^2 & \alpha = 1. \end{cases} \tag{39}$$

By using Liouvil derivative operator, Taylor expansion formula can be extended to continuous Taylor integral form:

$$f(x) = \int_0^\infty \frac{f^\alpha(x_0)}{(-1)^\alpha \Gamma(\alpha+1)} (x-x_0)^\alpha d\alpha \tag{40}$$

Expand the Taylor integral for $(k-p)^\beta$ at $k=0$:

$$(k-p)^\beta = \int_0^\infty \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)\Gamma(\alpha+1)} \frac{\sin(\pi\beta)}{\sin[\pi(\beta-\alpha)]} (-p)^{\beta-\alpha} k^\alpha d\alpha \tag{41}$$

Linear expansion of non-integer order power function

where

$$\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)\Gamma(\alpha+1)}\frac{\sin(\pi\beta)}{\sin[\pi(\beta-\alpha)]} \tag{42}$$

$$=\frac{\Gamma(\beta+1)\Gamma(\beta-\alpha)\Gamma(1-\beta+\alpha)}{\Gamma(\beta+1-\alpha)\Gamma(\alpha+1)\Gamma(\beta)\Gamma(1-\beta)} \tag{43}$$

$$=\frac{\beta\Gamma(1-\beta+\alpha)}{(\beta-\alpha)\Gamma(\alpha+1)\Gamma(1-\beta)} \tag{44}$$

As $\Gamma(x)$ has one order singularity at 0 and minus integer, the above equation has one order singularity at $\beta-\alpha=n$,n is integer and $n>-1$. $\beta\in(1,2]$, the singularity is at $\alpha=\beta,\beta-1$.

Linear expansion of non-integer order power function

The integrand $f(\alpha)$ is:

$$\frac{\Gamma(0)\sin(\pi\beta)}{\pi}k^{\beta}, \quad \alpha=\beta, \tag{45}$$

$$\frac{\Gamma(0)\sin(\pi\beta)}{\pi}(-\beta p)k^{\beta-1}, \quad \alpha=\beta-1. \tag{46}$$

The integrated result is:

$$(k-p)^{\beta}=k^{\beta}-\beta pk^{\beta-1} \tag{47}$$

For higher order β, this integral will give more one order singularities, namely more terms in the expanded formula.