Friedrichs-Lee model and Bound states, virtual states, resonances in hadron physics

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Motivation

- Hadronic states: Mesons, $q\bar{q}$, Baryons, qqq, ...
- The intermediate states in the scatterings: Resonance, virtual state(anti-bound), bound states.
- The intermediate state could be: $|q\bar{q}\rangle + |\text{two hadrons}\rangle \cdots$? E.g. $D\bar{D}^* \to \chi_{c1} \to D\bar{D}^*$.
- Pure composite states: dynamically generated. How to express them using the component states?
- Can we define the compositeness and elementariness for a state?
- Dynamically generated state: How is it generated from interaction?
- To study these theoretical problems, look at a solvable model is instructive: Friedrichs-Lee model.

THE SIMPLEST FRIEDRICHS MODEL [Friedrichs, Commun. Pure Appl. Math.,1(1948),361, See O. Civitaresea, M. Gadella, Phys.Rep.396,41 for review]

Different models in the same spirit: Lee model [PR,95,1329(1954)], Anderson model [PR,124,41(1961)], Jaynes Cummings, ...

$$H = H_0 + V$$

- Free Hamiltonian:bare discrete state $|1\rangle$, a continuum state $|\omega\rangle$, (set threshold=0 for simplicity)

$$H_0 = \omega_0 |1\rangle\langle 1| + \int_0^\infty \omega |\omega\rangle\langle\omega| \mathrm{d}\omega$$

- Interaction vertex:

$$V = \lambda \int_0^\infty [f(\omega)|\omega\rangle\langle 1| + f^*(\omega)|1\rangle\langle\omega|]\mathrm{d}\omega$$

- Orthonormal condition: $\langle 1|1\rangle = 1$, $\langle 1|\omega\rangle = 0$, and $\langle\omega|\omega'\rangle = \delta(\omega - \omega')$

  Completeness: $|1\rangle\langle 1| + \int_0^\infty d\omega|\omega\rangle\langle\omega| = 1$

This model is exactly solvable.

Eigenvalue equation:

$$H|\Psi(E)\rangle = (H_0 + V)|\Psi\rangle = E|\Psi(E)\rangle.$$

Solutions:

- Continuum: Eigenvalue $E > 0$, real
  Solution: define inverse resolvent

$$\eta^{\pm}(E) = E - \omega_0 - \lambda^2 \int_0^\infty \frac{f(\omega)f^*(\omega)}{E - \omega \pm i\epsilon} d\omega$$

$$|\Psi_{\pm}(E)\rangle \quad = \quad |E\rangle + \lambda \frac{f^*(E)}{\eta^{\pm}(E)} \left[|1\rangle + \lambda \int_0^\infty \frac{f(\omega)}{E - \omega \pm i\epsilon} |\omega\rangle d\omega \right]$$

- S-matrix:

$$S(E, E') = \delta(E - E') \left(1 - 2\pi i \frac{\lambda f(E) f^*(E)}{\eta^+(E)} \right).$$

- Discrete states:The zero point of $\eta(E)$ corresponds to eigenvalues of the full Hamiltonian — discrete states.

Discrete state solutions: Bound states

$$\eta^I(E) = E - \omega_0 - \lambda^2 \int_0^\infty \frac{f(\omega) f^*(\omega)}{E - \omega} \mathrm{d}\omega = 0$$

- Bound states: solutions on the first sheet real axis below the threshold.

$$|z_B\rangle = N_B \Big(|1\rangle + \lambda \int_0^\infty \frac{f(\omega)}{z_B - \omega} |\omega\rangle \mathrm{d}\omega \Big)$$

  where $N_B = (\eta'(z_B))^{-1/2} = (1 + \lambda^2 \int d\omega \frac{|f(\omega)|^2}{(z_B - \omega)^2})^{-1/2}$, such that $\langle z_B | z_B \rangle = 1$.

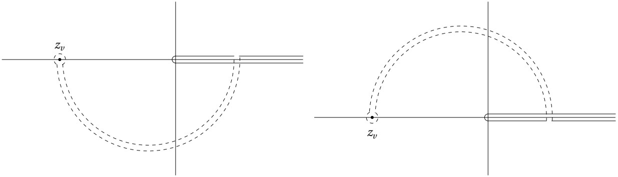
- Elementariness: $Z = N_B^2$;
  Compositeness: $X = N_B^2 \lambda^2 \int d\omega \frac{|f(\omega)|^2}{(z_B - \omega)^2}$.
- Eg. If $\omega_0 < 0$, there could be a bound state. In the weak coupling limit, it $\to |1\rangle$,
- Eg. there could also be dynamically generated bound state when the coupling is strong.

Discrete state solutions:Virtual states

- Virtual states: Solutions on the second sheet real axis below the threshold.

$$|z_v^{\pm}\rangle = N_v^{\pm}\Big(|1\rangle + \lambda \int_0^{\infty} \frac{f(\omega)}{[z_v-\omega]_{\pm}}|\omega\rangle \mathrm{d}\omega\Big), \quad \langle \tilde{z}_v^{\pm}| = \langle z_v^{\mp}|\,,$$

  where
  $N_v^{-} = N_v^{+*} = (\eta'^{+}(z_v))^{-1/2} = (1+\lambda^2 \int d\omega \frac{|f(\omega)|^2}{[(z_v-\omega)_+]^2})^{-1/2}$,
  such that $\langle \tilde{z}_v^{\pm}|z_v^{\pm}\rangle = 1$. No probability explanation.
- Elementariness & compositeness not well-defined.

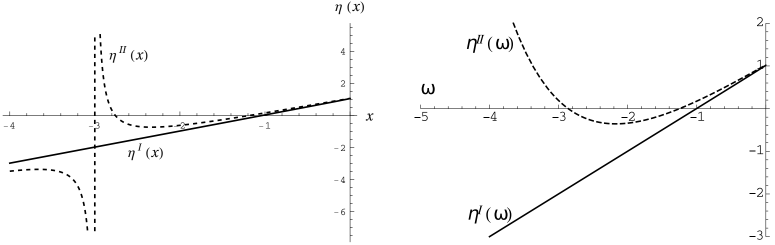


Discrete state solutions:Virtual states

- When $\omega_0 < 0$, a bound state generated from $|1\rangle$ is always accompanied with a virtual state for weak coupling, $\to |1\rangle$.
- Virtual states from the singularity of the vertex function, $(|z_v\rangle \not\to |1\rangle$, at $\lambda \to 0)$

$$\eta^I = z - \omega_0 - \lambda^2 \int_0^\infty \frac{|f(\omega)|^2}{z-\omega} \mathrm{d}\omega, \quad (G(\omega) \equiv |f(\omega)|^2)$$

$$\eta^{II}(\omega) = \eta^I(\omega) + 2\pi i \lambda^2\, G^{II}(\omega) = \eta^I(\omega) - 2\lambda^2 \pi i\, G(\omega),$$

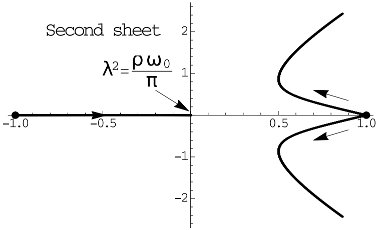


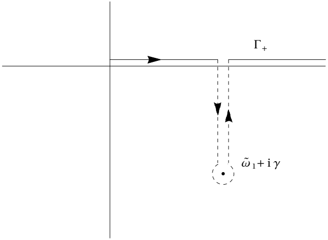
$G(\omega) \sim \frac{\sqrt{\omega}}{\omega+3}$ $\qquad$ $G(\omega) \sim \sqrt{\omega}\, e^{-\omega/\Lambda}$,

Discrete state solutions: Resonance

- Resonant states: $\omega_0 >$ threshold, the discrete state becomes a pair of solutions z_R, z_R^*, on the second sheet of the complex plane. $\hat{H}|z_R\rangle = z_R|z_R\rangle$

$$|z_R\rangle = N_R\Big(|1\rangle + \lambda \int_0^\infty \mathrm{d}\omega \frac{f(\omega)}{[z_R - \omega]_+}|\omega\rangle\Big),$$
$$|z_R^*\rangle = N_R^*\Big(|1\rangle + \lambda \int_0^\infty \mathrm{d}\omega \frac{f(\omega)}{[z_R^* - \omega]_-}|\omega\rangle\Big),$$





Discrete state solutions: Resonance

Resonant states:

- Normalization: $\langle z_R | z_R \rangle = 0$, naïve argument, $z_R^* \neq z_R$,

$$\langle z_R | \hat{H} | z_R \rangle = z_R \langle z_R | z_R \rangle = z_R^* \langle z_R | z_R \rangle = 0$$

  $|z_R\rangle$ is not in the Hilbert space — need rigged Hilbert space description.

- Left eigenstates:$\langle \tilde{z}_R | \hat{H} = \langle \tilde{z}_R | z_R$

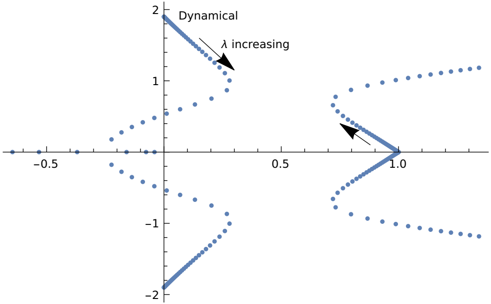
$$\langle \tilde{z}_R | = \langle z_R^* | = N_R \Big(\langle 1 | + \lambda \int_0^\infty \mathrm{d}\omega \frac{f(\omega)}{[z_R - \omega]_+} \langle \omega | \Big),$$
$$\langle \tilde{z}_R^* | = \langle z_R | = N_R^* \Big(\langle 1 | + \lambda \int_0^\infty \mathrm{d}\omega \frac{f(\omega)}{[z_R^* - \omega]_-} \langle \omega | \Big).$$

  N_R is a complex normalization parameter, $N_R = (\eta'^+(z_R))^{-1/2} = (1 + \lambda^2 \int d\omega \frac{|f(\omega)|^2}{[(z_R - \omega)_+]^2})^{-1/2}$ such that $\langle \tilde{z}_R | z_R \rangle = 1$, [Sekihara,Hyodo,Jido,PTEP 2015 (2015) 063D04]

- Other physical proposal of “elementariness” and “compositeness”: [Guo,Oller,PRD93,096001].

Discrete state solutions: Dynamically generated Resonance

Dynamical resonance generated from the singularity of the vertex function.

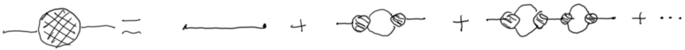


$$G(\omega) \sim \frac{\sqrt{\omega}}{\omega^2 + a^2},\ a = 1.9,\ \omega_0 = 1$$

- $G = \sqrt{\omega} e^{-\omega^2/a^2}$ case : Similar situation could happen.
- A caveat to using form factor put by hand to suppress the high E contribution: The form factor may play an important role in generating the dynamical state.

Bubble chain sum

- What Friedrichs model does:

$\eta^{-1}(\omega) \sim$ 

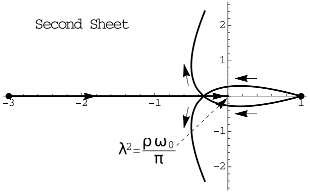
  Similar to the UQM by Tönqvist [Z. Phys. C 68, 647 (1995)]

- Effective field theory bubble chain: Similar results about the poles.

Other interesting things

- Higher order poles: [A. Mondragon and E Hernandez,J.Phy.A26(1993),5595;A. Bohm et.al.JMP38(1997),6072; I. E. Antoniou et.al.,JMP39(1997),2459; E. Hernández et.al., Int.J.Theo.Phys.,42(2003), 2167]

  Hamiltonian:
  can not be diagonalized exactly,
  $\rightarrow$ Jordan form



- Completeness relation: redefine the continuum states to including the resonances into the completeness relation [T. Petrosky et..al. Phys.A173(1991),175;ZX,Zhou,PRD94(2016)076006]

GENERALIZATION: [ZYZ&ZX,JMP.58(2017),062110;JMP58(2017), 072102]

Real world: interaction between $|0; JM\rangle$ and $|\mathbf{p}_1\mathbf{p}_2, S\rangle$

- Partial wave decomposition: $|\mathbf{p}_1\mathbf{p}_2\rangle \to |p, JM, lS\rangle \sim |\omega, l\rangle$

$$H = M_0|0\rangle\langle 0| + \sum_l \int \mathrm{d}\omega\, \omega|\omega, l\rangle\langle\omega, l| + \sum_l \int d\omega\, g_l(\omega)|0\rangle\langle\omega, l| + h.c.$$

- Include more discrete states and more continuum states: No direct coupled continuum channel.
- Separable interaction potential like in [E. Hernández et.al, PRC29(1984),722;Aceti et.al., PRD86,(2012),014012;Sekihara, PTEP(2015)063D04;Weinberg,PR131(1963),441;...]: solvable.

$$H = \sum_{i=1}^{D} M_i|i\rangle\langle i| + \sum_{i=1}^{C}\int_{a_i}^{\infty} \mathrm{d}\omega\, \omega|\omega; i\rangle\langle\omega; i|$$
$$+ \sum_{i,j=1}^{C} v_{ij}\Big(\int_{a_i}^{\infty}\mathrm{d}\omega f_i(\omega)|\omega; i\rangle\Big)\Big(\int_{a_j}^{\infty}\mathrm{d}\omega f_j^*(\omega)\langle\omega; j|\Big)$$
$$+ \sum_{j=1}^{D}\sum_{i=1}^{C}\Big[u_{ji}^*|j\rangle\Big(\int_{a_i}^{\infty}\mathrm{d}\omega f_i^*(\omega)\langle\omega; i|\Big) + u_{ji}\Big(\int_{a_i}^{\infty}\mathrm{d}\omega f_i(\omega)|\omega; i\rangle\Big)\langle j|\Big]$$

- Relativistic Friedrichs-Lee model.

Dynamically generated states

Study the near threshold behavior of the dynamically generated states.

- No discrete bare states $\to$ dynamically generated discrete state — Bound state (molecular state), resonances, or virtual state.
- Hamiltonian:

$$H = \int_a \mathrm{d}\omega\, \omega |\omega\rangle\langle\omega| \pm \lambda^2 \int_a \mathrm{d}\omega \int_a \mathrm{d}\omega' f(\omega) f^*(\omega') |\omega\rangle\langle\omega'| \quad (1)$$

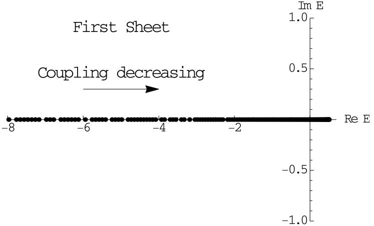
- Vertex function $f(\omega) = (\omega - a)^{(l+1/2)/2} \exp\{-(\omega - a)/(2\Lambda)\}$.
- Discrete state pole position:

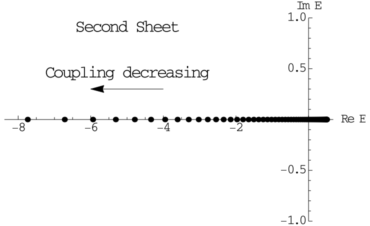
$$\mathbf{M}_\pm(E) = \det\, M_\pm = 1 \pm \lambda^2 G(E) = 1 \pm \lambda^2 \int_a \mathrm{d}\omega \frac{|f(\omega)|^2}{\omega - E} = 0$$

- $-$ sign: attractive.

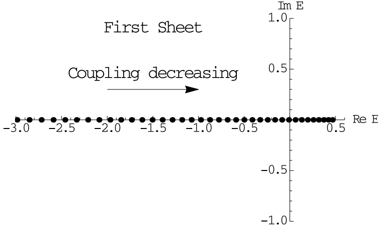
Eg: dynamically generated states, attractive potential

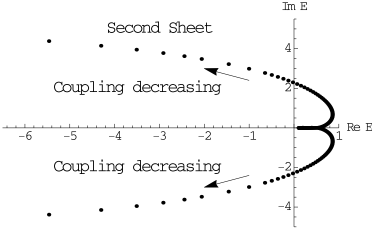
- S-wave: Strong coupling, a bound state $\rightarrow$ Weak coupling, a virtual state





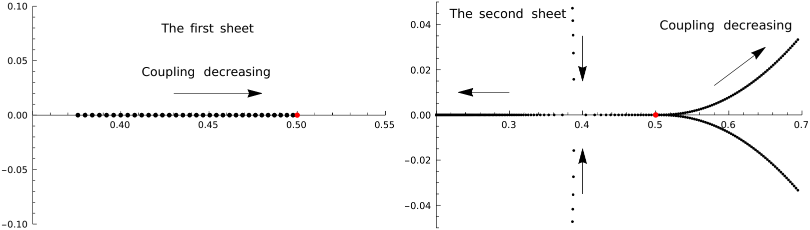
- P-wave: Strong coupling, A bound state and a virtual state $\rightarrow$ Weak coupling, a pair of resonance poles.





Eg: D-wave dynamically generated states

Attractive coupling:



- The resonance poles merge at the threshold, and one becomes a virtual state, the other becomes a bound states.

Near threshold poles for attractive potential: when coupling is becoming stronger

- $l \geq 1$: a virtual state and a bound state appear together.
- $l = 0$, one bound/virtual state near the threshold.
- Explained using the effective range expansion in [Taylor, Scattering Theory; Hanhart et. al. PLB739(2014)375] and also using Jost function in [Hyodo,PRC90:055208(2014);]

Dynamical v.s. Elementary

Elementary: originated from the bare discrete state
Dynamical: generated by interaction

- S-wave bound state:
  Dynamical state: can have no acompanied virtual state.
  Elementary state: always accompanied with a virtual state pole at weak coupling
  — Pole counting rule [D. Morgan, NPA543(1992),632;Baru et.al., Phys. Lett. B586(2004),53;Ou Zhang,et.al., Phys.Lett.B680(2009),453]
- Higher partial wave , no such a difference: The dynamically generated state if appears from the threshold (resonance pole merging), it must acompanied with a virtual state.
- In weak coupling limit: The dynamically generated states do not go to bare states, but towards the singular point of the form factor.

Thoughts on Elementariness and Compositeness

Whether we need the elementariness and compositeness for the resonance:

- As long as it is unstable it is non-normalizable: The elementariness and compositeness are not well-defined. Narrow resonance, approximately defined.
- They are not physical observable: No experiment can really prepare a pure resonance. Resonances only appear as the intermediate states in the scattering.
- The only observable is the scattering cross section or the events.

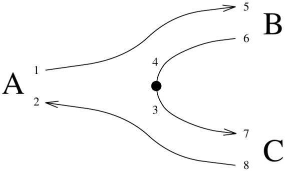
Friedrichs-QPC scheme

To study to hadron spectrum using nonrelativistic Friedrichs model: Solve $\det[\eta(E)] = 0$.

- Coupling vertex between the discrete state and continuum $f(\omega)$: dynamically given.
- The interactions can be estimated using differnent models: we will use the QPC (3P0) model.

$$\langle BC|T|A\rangle = \delta^3(\mathbf{P_f} - \mathbf{P_i})M^{ABC}$$

$$T = -3\gamma\sum_m \langle 1m1-m|00\rangle \int d^3\mathbf{p_3}\, d^3\mathbf{p_4}\,\delta^3(\mathbf{p_3}+\mathbf{p_4}) \times \mathcal{Y}_1^m\left(\frac{\mathbf{p_3}-\mathbf{p_4}}{2}\right)\chi_{1-m}^{34}\phi_0^{34}\omega_0^{34}b_3^\dagger(\mathbf{p_3})d_4^\dagger(\mathbf{p_4}).$$



γ: the strength of creating a quark-antiquark pair.
[Blundell,Godfrey,PRD53(1996),3700]

- The bare mass and wave functions of A, B, C are GI's results. [Godfrey & Isgur, PRD32,189(1985)].

Effect: Including the hadron-hadron interactions into the GI model.

Application: 1^{--} charmonium

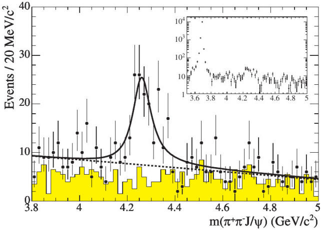
We consider the experimental data in $\sqrt{s} \sim 3.73 - 5$GeV from
$e^+e^- \to D\bar{D}$[PRD77,011103(2008)],
$e^+e^- \to D\bar{D}^*$, $D^*\bar{D}^*$[PRD97,012002(2018)],
$e^+e^- \to D\bar{D}\pi$[PRL100,062001(2008)],

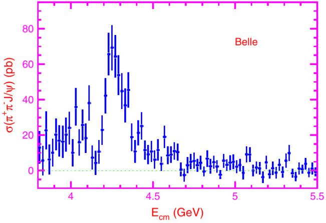
- $\psi(3770)$, $\psi(4040)$, $\psi(4160)$, $\psi(4415)$ are well established: $\psi(1^3D_1)$, $\psi(3^3S_1)$, $\psi(2^3D_1)$, $\psi(4^3S_1)$, considered as bare discrete state.
- $\psi(4230)$, $\psi(4360)$, $\psi(4660)$ mainly appear in hidden-charm channels, not included.
  $\psi(4230) \to \pi^+\pi^- J/\psi$,
  $\psi(4360), \psi(4660) \to \pi^+\pi^-\psi(3686)$

$\psi(4230)$: 1^{--}

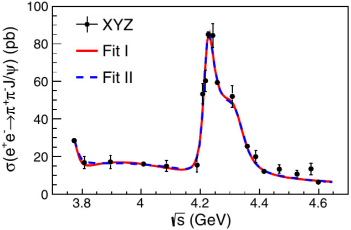
- Was observed in $e^+e^- \to \pi^+\pi^- J/\psi$:

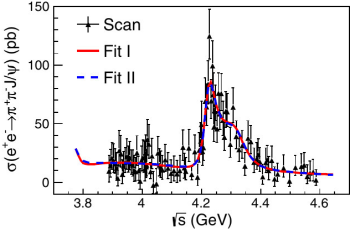
  [BaBar,PRL95(2005),142001;Belle,PRL99(2007),182004]





- BES III: [PRL 118, 092001 (2017)]



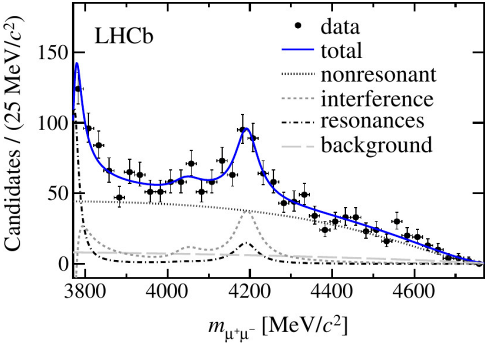


  two states $\sqrt{s} = 4222.0$MeV, $\Gamma = 44$MeV; $\sqrt{s} = 4320.0$MeV, $\Gamma = 101$MeV.

- Only appears in the hidden charm channels.

$\psi(4160)$

- First observed by The DASP collaboration, in analyzing the R value,[Phys. Lett. B 76, 361(1978)]
- The most recent PDG: LHCb[PRL 111, 112003 (2013)]
  $B \to \psi(4160) + K^+ \to \mu^+\mu^- K^+$
  $M = 4191 \pm 5$MeV,$\Gamma = 70 \pm 10$MeV.



- Mostly in the open charm channels.
- $\psi(4230)$ and $\psi(4160)$ are close to each other.

Our scheme

Friedrichs model + QPC model.

- Bare discretes $c\bar{c}$ states: GI wave function, with bare mass parameters to be fitted.
  A backgound bare $\psi(2S)$ state, $\psi(1D)$, $\psi(3S)$, $\psi(2D)$, $\psi(4S)$.
- Coupling to continuum states: Quark pair creation strength, γ, interaction vertices $f_{\rho i}(\omega)$
  Continua: $D\bar{D}, D\bar{D}^*, D^*\bar{D}^*, D\bar{D}_2^*(\bar{D}\pi)$
- Couplings to e^+e^-: $g_{e\rho}$, phase, $\phi_{e\rho}$

Results: $\chi^2/d.o.f \sim 379/(293-15) = 1.36$

Coupled channel Friedrichs model + QPC

Include more discrete states and more continuum states (No direct coupling between continuum channels)

$$V = \sum_{\rho=1}^{D}\sum_{i=1}^{C}\int_{a_i}^{\infty} \mathrm{d}E[f^*_{\rho i}(E)|\rho\rangle\langle E;i| + f_{\rho i}(E)|E;i\rangle\langle\rho|].$$

$$S_{i,j} = \delta_{ij} - 2\pi i\sum_{\rho,\lambda} f^*_{\rho j}(E)[\eta^{-1}]_{\rho\lambda} f_{\lambda i}(E),$$

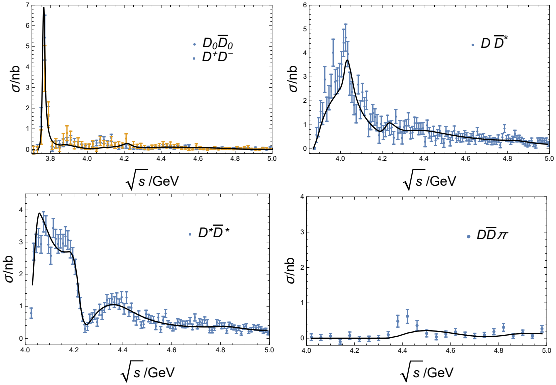
$$[\eta]_{\rho\lambda} \equiv (E - M_\rho)\delta_{\rho\lambda} - \sum_{i=1}^{C}\int_{a_i}^{\infty}\mathrm{d}E'\frac{f^*_{\rho i}(E')f_{\lambda i}(E')}{(E-E')}.$$

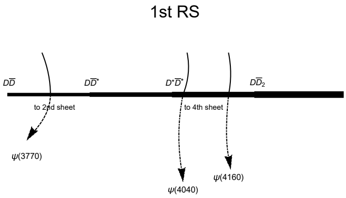
Automatically satisfies the analyticity and coupled channel Unitarity. Discrete State poles:

$$\det[\eta] = 0$$

Scattering cross section:

$$\sigma_i(E) = \frac{32\pi^5\alpha}{E^5}|\sum_{\rho\lambda} g_{e\rho}[\eta^{-1}]_{\rho\lambda} f_{\lambda i}|^2,$$





Pole positions:

$$z_0^{II} = 3.762 - \frac{0.020}{2} i\text{GeV},$$

$$z_{0,1}^{IV} = 4.028 - \frac{0.068}{2} i\text{GeV},$$

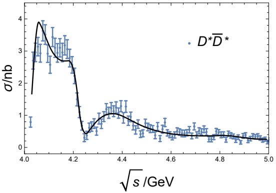
$$z_{0,2}^{IV} = 4.222 - \frac{0.064}{2} i\text{GeV}.$$

PDG:

$$\psi(3770): M = 3773.7 \pm 0.4\text{MeV}, \Gamma = 27.2 \pm 1\text{MeV}.$$

$$\psi(4040): M = 4039 \pm 1\text{MeV}, \Gamma = 80 \pm 10\text{MeV}.$$

$$\psi(4160): M = 4190 \pm 5\text{MeV}, \Gamma = 70 \pm 10\text{MeV}.$$



- $\psi(4160)$: pole at $z^{IV}_{0,2} = 4.222 - \frac{0.064}{2} i\text{GeV}$.
- Experiments: parameterized using Breit-Wigners
  - Resonances with interference: Multiple solution
  - Violation of unitarity: important for interfering resonances
- The pole position for $\psi(4160)$ is close to the mass for $\psi(4230)$: not at the hill top , but at the half hillside.

  $\psi(4230)$: $M = 4222.0$MeV, $\Gamma = 44$MeV
  Appears in weakly coupled channels.
- Conjecture: $\psi(4160)$ and $\psi(4230)$ could be the same state. This may explain why $\psi(4160)$ mostly appears in open-charm channels but $\psi(4230)$ in hidden-charm channels.
- Needs the R-value and other analysis to confirm.

Summary

- As an rigourously solvable model, Friedrichs model helps us in understanding the resonances, virtual states, and bound states
- Resonances, virtual states: not normalizable as usual, compositeness and elementariness not well-defined.
- How dynamical state is generated from the interaction between the discrete state and the continuum.
- Friedrichs-QPC Scheme: interaction vertices from QPC, Satisfying the unitary and analyticity. Can be used in real hadronic physics.
- $\psi(4160)$ and $\psi(4230)$ could be the same state: different appearance in the strong interfering channels and weak interaction channels.

Thank you !

Backup

Fit parameters: results

	Background	$\psi(1D)$	$\psi(3S)$	$\psi(2D)$	$\psi(4S)$
Bare mass	1.83	4.09	4.36	4.62	5.35
$g_{e\rho}$	26.4	1.82	5.32	-3.68	-6.12
$\phi_{e\rho}/^{\circ}$	0(fixed)	35.2	-31.6	64.6	129.3
γ	4.48				