量子场论多圈图解析计算

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Contents

I. Introduction

1. Background

- Higher-order radiative corrections are more important, with the increasing precision of measurements at the future colliders: CLIC, ILC, CEPC, FCC, HL-LHC $\cdots$
- One-loop Feynman integrals are well known analytically in the time-space dimension $D = 4 - 2\varepsilon$.
  However, how to perform analytically multi-loop Feynman integrals is still a challenge.
- **Considering Feynman integrals as the generalized hypergeometric functions, one finds that the $D-$module of a Feynman diagram is isomorphic to Gel'fand-Kapranov-Zelevinsky (GKZ) $D-$module.**

I. Introduction

2. Relevant research

- Hypergeometric functions of some Feynman integrals are obtained from Mellin-Barnes representations.

  Feng, Chang, Chen, Gu, Zhang, *NPB* 927(2018)516 [arXiv:1706.08201]
  Feng, Chang, Chen, Zhang, *NPB* 940(2019)130 [arXiv:1809.00295]
  Gu, Zhang, *CPC* 43(2019)083102 [arXiv:1811.10429]
  Gu, Zhang, Feng, *IJMPA* 35(2020)2050089.

- Using GKZ hypergeometric system, we can obtain the fundamental solution systems of Feynman integrals.

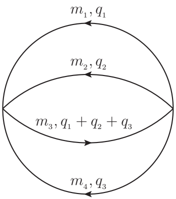
  Feng, Chang, Chen, Zhang, *NPB* 953(2020)114952, [arXiv:1912.01726]
  Feng, Zhang, Chang, *PRD* 106(2022)116025 [arXiv: 2206.04224]
  Feng, Zhang, Dong, Zhou, *EPJC* 83(2023)314 [arXiv:2209.15194].
  Zhang, Feng, *JHEP* 05(2023)075 [arXiv: 2303.02795].

I. Introduction

3. Generally strategy

- **We can derive GKZ hypergeometric systems of Feynman integrals, basing on Mellin-Barnes representations and Miller's transformation. We can formulate Feynman integrals as hypergeometric functions on general compact manifold or Grassmannian manifold.**
- Steps: (1) we write out the GKZ hypergeometric systems satisfied by the Feynman integrals on general compact manifold or proper Grassmannian manifold $G_{k,n}$. (2) fundamental solution systems are constructed in neighborhoods of regular singularities of the GKZ hypergeometric systems. The combination coefficients can be determined from Feynman integrals with some special kinematic parameters.

II. 3-loop vacuum on compact manifold



- Feynman integral of the 3-loop vacuum diagram with 4 propagates is written as

$$U_4 = \left(\Lambda^2_{\rm RE}\right)^{6-\frac{3D}{2}} \int \frac{d^D q_1}{(2\pi)^D} \frac{d^D q_2}{(2\pi)^D} \frac{d^D q_3}{(2\pi)^D} \times \frac{1}{(q_1^2 - m_1^2)(q_2^2 - m_2^2)((q_1 + q_2 + q_3)^2 - m_3^2)(q_3^2 - m_4^2)} . \quad (2.1)$$

- Zhang, Feng, GKZ hypergeometric systems of the three-loop vacuum Feynman integrals, *JHEP* 05 (2023) 075 [arXiv: 2303.02795].

II. 3-loop vacuum on compact manifold

- Through Mellin-Barnes transformation

$$U_4 = \frac{\left(\Lambda^2_{\rm RE}\right)^{6-\frac{3D}{2}}}{(2\pi i)^3} \int_{-i\infty}^{+i\infty} ds_1 ds_2 ds_3 \left[\prod_{i=1}^{3} (-m_i^2)^{s_i} \Gamma(-s_i)\Gamma(1+s_i)\right] I_q \,, \quad (2.2)$$

where

$$I_q \equiv \int \frac{d^D q_1}{(2\pi)^D} \frac{d^D q_2}{(2\pi)^D} \frac{d^D q_3}{(2\pi)^D} \frac{1}{(q_1^2)^{1+s_1} (q_2^2)^{1+s_2} ((q_1+q_2+q_3)^2)^{1+s_3} (q_3^2 - m_4^2)} \,. \quad (2.3)$$

II. 3-loop vacuum on compact manifold

- Using Feynman parametrization and Beta function,

$$B(m,n) = \int_0^1 dx\, x^{m-1}(1-x)^{n-1} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\ , \tag{2.4}$$

one can have

$$I_q = \frac{-i}{(4\pi)^{\frac{3D}{2}}}(-)^{\sum_{i=1}^{3} s_i}\left(\frac{1}{m_4^2}\right)^{4-\frac{3D}{2}+\sum_{i=1}^{3} s_i}\left[\prod_{i=1}^{3}\Gamma(\frac{D}{2}-1-s_i)\Gamma(1+s_i)^{-1}\right]$$
$$\times\Gamma(3-D+\sum_{i=1}^{3} s_i)\Gamma(4-\frac{3D}{2}+\sum_{i=1}^{3} s_i)\ . \tag{2.5}$$

II. 3-loop vacuum on compact manifold

- Mellin-Barnes representation of the Feynman integral:

$$U_4 = \frac{-im_4^4}{(2\pi i)^3(4\pi)^6}\left(\frac{4\pi\Lambda_{\rm RE}^2}{m_4^2}\right)^{6-\frac{3D}{2}}\int_{-i\infty}^{+i\infty} ds_1 ds_2 ds_3 \left[\prod_{i=1}^{3}\left(\frac{m_i^2}{m_4^2}\right)^{s_i}\Gamma(-s_i)\right]$$
$$\times\left[\prod_{i=1}^{3}\Gamma(\frac{D}{2}-1-s_i)\right]\Gamma(3-D+\sum_{i=1}^{3}s_i)\Gamma(4-\frac{3D}{2}+\sum_{i=1}^{3}s_i)\ . \quad (2.6)$$

- It is well known that negative integers and zero are simple poles of the function $\Gamma(z)$. As all s_i contours are closed to the right in corresponding complex planes, one finds that the analytic expression of the the three-loop vacuum integral can be written as the linear combination of generalized hypergeometric functions.

II. 3-loop vacuum on compact manifold

- Taking the residue of the pole of $\Gamma(-s_i),\ (i=1,\ 2,\ 3)$, we can derive one linear independent term:

$$U_4 \ni \frac{-im_4^4}{(4\pi)^6}\left(\frac{4\pi\Lambda_{\rm RE}^2}{m_4^2}\right)^{6-\frac{3D}{2}}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}(-)^{\sum_{i=1}^{3}n_i}x_1^{n_1}x_2^{n_2}x_3^{n_3}$$
$$\times\left[\prod_{i=1}^{3}\Gamma(\frac{D}{2}-1-n_i)(n_i!)^{-1}\right]\Gamma(3-D+\sum_{i=1}^{3}n_i)$$
$$\times\Gamma(4-\frac{3D}{2}+\sum_{i=1}^{3}n_i)\ , \qquad (2.7)$$

with $x_i=\frac{m_i^2}{m_4^2},\ (i=1,\ 2,\ 3)$.

II. 3-loop vacuum on compact manifold

$$U_4 \ni \frac{im_4^4}{(4\pi)^6}\left(\frac{4\pi\Lambda_{\mathrm{RE}}^2}{m_4^2}\right)^{6-\frac{3D}{2}} \frac{\pi^3}{\sin^3\frac{\pi D}{2}} T_4(\mathbf{a},\ \mathbf{b}\ \big|\ \mathbf{x})\ , \tag{2.8}$$

with

$$T_4(\mathbf{a},\ \mathbf{b}\ \big|\ \mathbf{x}) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} A_{n_1 n_2 n_3} x_1^{n_1} x_2^{n_2} x_3^{n_3}\ , \tag{2.9}$$

$$A_{n_1 n_2 n_3} = \frac{\Gamma(a_1 + \sum_{i=1}^{3} n_i)\Gamma(a_2 + \sum_{i=1}^{3} n_i)}{n_1!n_2!n_3!\Gamma(b_1+n_1)\Gamma(b_2+n_2)\Gamma(b_3+n_3)}\ . \tag{2.10}$$

where $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2, b_3)$ with

$$a_1 = 3 - D,\ a_2 = 4 - \frac{3D}{2},\ b_1 = b_2 = b_3 = 2 - \frac{D}{2}\ . \tag{2.11}$$

II. 3-loop vacuum on compact manifold

- We can define auxiliary function

$$\Phi_4(\mathbf{a},\ \mathbf{b}\ \Big|\ \mathbf{x},\ \mathbf{u},\ \mathbf{v}) = \mathbf{u}^{\mathbf{a}}\mathbf{v}^{\mathbf{b}-\mathbf{e}_3}T_4(\mathbf{a},\ \mathbf{b}\ \Big|\ \mathbf{x})\ . \qquad (2.12)$$

Through Miller's transformation,

$$\begin{aligned}\vartheta_{u_j}\Phi_4(\mathbf{a},\ \mathbf{b}\ \Big|\ \mathbf{x},\ \mathbf{u},\ \mathbf{v}) &= a_j\Phi_4(\mathbf{a},\ \mathbf{b}\ \Big|\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})\ ,\\ \vartheta_{v_k}\Phi_4(\mathbf{a},\ \mathbf{b}\ \Big|\ \mathbf{x},\ \mathbf{u},\ \mathbf{v}) &= (b_k-1)\Phi_4(\mathbf{a},\ \mathbf{b}\ \Big|\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})\ ,\end{aligned} \qquad (2.13)$$

which naturally induces the notion of GKZ hypergeometric system. Euler operators: $\vartheta_{x_k} = x_k\partial_{x_k}$.

II. 3-loop vacuum on compact manifold

- Through the transformation

$$z_j = \frac{1}{u_j}, \quad z_{2+k} = v_k, \; z_{5+k} = \frac{x_k}{u_1 u_2 v_k}, \qquad (2.14)$$

we have GKZ hypergeometric system for the integral

$$\mathbf{A_4} \cdot \vec{\vartheta}_4 \Phi_4 = \mathbf{B_4} \Phi_4 \;, \qquad (2.15)$$

$$\mathbf{A_4} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix},$$

$$\vec{\vartheta}_4^{\,T} = (\vartheta_{z_1}, \; \cdots \;, \; \vartheta_{z_8}) \;,$$

$$\mathbf{B_4}^T = (-a_1, \; -a_2, \; b_1 - 1, \; b_2 - 1, \; b_3 - 1) \;. \qquad (2.16)$$

II. 3-loop vacuum on compact manifold

- Correspondingly the dual matrix $\tilde{\mathbf{A}}_4$ of $\mathbf{A}_4$ is

$$\tilde{\mathbf{A}}_4 = \begin{pmatrix} -1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}. \qquad (2.17)$$

  The row vectors of the matrix $\tilde{\mathbf{A}}_4$ induce the integer sublattice $\mathbf{B}$ which can be used to construct the formal solutions in hypergeometric series.

- Actually the integer sublattice $\mathbf{B}$ indicates that the solutions of the system should satisfy the equations in Eq. (2.15) and the following hyperbolic equations simultaneously

$$\frac{\partial^2\Phi_4}{\partial z_1\partial z_2} = \frac{\partial^2\Phi_4}{\partial z_3\partial z_6}\,, \quad \frac{\partial^2\Phi_4}{\partial z_1\partial z_2} = \frac{\partial^2\Phi_4}{\partial z_4\partial z_7}\,, \quad \frac{\partial^2\Phi_4}{\partial z_1\partial z_2} = \frac{\partial^2\Phi_4}{\partial z_5\partial z_8}\,. \qquad (2.18)$$

II. 3-loop vacuum on compact manifold

- Defining the combined variables

$$y_1 = \frac{z_3 z_6}{z_1 z_2} \,, \quad y_2 = \frac{z_4 z_7}{z_1 z_2} \,, \quad y_3 = \frac{z_5 z_8}{z_1 z_2} \,, \tag{2.19}$$

we write the solutions as

$$\Phi_4(\mathbf{z}) = \Big(\prod_{i=1}^{8} z_i^{\alpha_i} \Big) \varphi_4(y_1, \, y_2, \, y_3) \,. \tag{2.20}$$

Here $\vec{\alpha}^{\,T} = (\alpha_1, \, \alpha_2, \, \cdots \, , \, \alpha_8)$ denotes a sequence of complex number such that

$$\mathbf{A_4} \cdot \vec{\alpha} = \mathbf{B_4} \,, \tag{2.21}$$

namely,

$$\alpha_1 + \alpha_6 + \alpha_7 + \alpha_8 = -a_1 \,, \quad \alpha_2 + \alpha_6 + \alpha_7 + \alpha_8 = -a_2 \,,$$
$$\alpha_3 - \alpha_6 = b_1 - 1 \,, \quad \alpha_4 - \alpha_7 = b_2 - 1 \,, \quad \alpha_5 - \alpha_8 = b_3 - 1 . \tag{2.22}$$

II. 3-loop vacuum on compact manifold

- To construct the hypergeometric series solutions of the GKZ hypergeometric system in Eq. (2.15) together with corresponding hyperbolic equations in Eq. (2.18) through triangulation is equivalent to choose a set of the linear independent column vectors of the matrix in Eq. (2.17) which spans the dual space.
- We denote the submatrix composed of the first, third, and fourth column vectors of the dual matrix of Eq. (2.17) as $\tilde{\mathbf{A}}_{134}$, i.e.

$$\tilde{\mathbf{A}}_{134} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} . \quad (2.23)$$

II. 3-loop vacuum on compact manifold

- Obviously $\det \tilde{\mathbf{A}}_{134} = -1 \neq 0$, and

$$\begin{aligned}\mathbf{B}_{134} &= \tilde{\mathbf{A}}_{134}^{-1} \cdot \tilde{\mathbf{A}}_{4} \\ &= \left(\begin{array}{cccccccc} 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \end{array}\right) . \end{aligned} \quad (2.24)$$

  Taking 3 row vectors of the matrix $\mathbf{B}_{134}$ as the basis of integer lattice, one constructs the GKZ hypergeometric series solutions in parameter space through choosing the sets of column indices $I_i \subset [1, 8]$ $(i = 1, \cdots, 8)$ which are consistent with the basis of integer lattice $\mathbf{B}_{134}$.

II. 3-loop vacuum on compact manifold

- We take the set of column indices $I_1 = [2, 5, 6, 7, 8]$, i.e. the implement $J_1 = [1, 8] \setminus I_1 = [1, 3, 4]$. The choice on the set of indices implies the exponent numbers $\alpha_1 = \alpha_3 = \alpha_4 = 0$. Through Eq. (2.22), one can have

$$\begin{aligned} &\alpha_2 = a_1 - a_2,\ \alpha_5 = b_1 + b_2 + b_3 - a_1 - 3, \\ &\alpha_6 = 1 - b_1,\ \alpha_7 = 1 - b_2,\ \alpha_8 = b_1 + b_2 - a_1 - 2\,. \end{aligned} \tag{2.25}$$

Combined with Eq. (2.11), we can have

$$\alpha_2 = \frac{D}{2} - 1,\ \alpha_5 = -\frac{D}{2},\ \alpha_6 = \frac{D}{2} - 1,\ \alpha_7 = \frac{D}{2} - 1,\ \alpha_8 = -1\,. \tag{2.26}$$

II. 3-loop vacuum on compact manifold

- According the basis of integer lattice $\mathbf{B}_{134}$, the corresponding hypergeometric series solution with triple independent variables is written as

$$\Phi^{(1)}_{[134]}(\alpha, z) = \prod_{i=1}^{8} z_i^{\alpha_i} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} c^{(1)}_{[134]}(\alpha, \mathbf{n}) \Big(\frac{z_1 z_2}{z_5 z_8}\Big)^{n_1} \Big(\frac{z_3 z_6}{z_5 z_8}\Big)^{n_2} \Big(\frac{z_4 z_7}{z_5 z_8}\Big)^{n_3}$$

$$= \prod_{i=1}^{8} z_i^{\alpha_i} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} c^{(1)}_{[134]}(\alpha, \mathbf{n}) \Big(\frac{1}{y_3}\Big)^{n_1} \Big(\frac{y_1}{y_3}\Big)^{n_2} \Big(\frac{y_2}{y_3}\Big)^{n_3} , \quad (2.27)$$

with the coefficient is

$$c^{(1)}_{[134]}(\alpha, \mathbf{n}) = \Big\{ n_1! n_2! n_3! \Gamma(1+\alpha_2+n_1)\Gamma(1+\alpha_5-n_1-n_2-n_3)$$
$$\times\Gamma(1+\alpha_6+n_2)\Gamma(1+\alpha_7+n_3)\Gamma(1+\alpha_8-n_1-n_2-n_3)\Big\}^{-1} . \quad (2.28)$$

II. 3-loop vacuum on compact manifold

- And then, through Eq. (2.26), the corresponding hypergeometric series solution can be written as

$$\Phi^{(1)}_{[134]}(\alpha, z) = y_1^{\frac{D}{2}-1} y_2^{\frac{D}{2}-1} y_3^{-1} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} c^{(1)}_{[134]}(\alpha, \mathbf{n}) \Big(\frac{1}{y_3}\Big)^{n_1} \Big(\frac{y_1}{y_3}\Big)^{n_2} \Big(\frac{y_2}{y_3}\Big)^{n_3} , \tag{2.29}$$

with the coefficient is

$$c^{(1)}_{[134]}(\alpha, \mathbf{n}) = \frac{\Gamma(\frac{D}{2} + n_1 + n_2 + n_3)\Gamma(1 + n_1 + n_2 + n_3)}{n_1! n_2! n_3! \Gamma(\frac{D}{2} + n_1)\Gamma(\frac{D}{2} + n_2)\Gamma(\frac{D}{2} + n_3)} . \tag{2.30}$$

Here, the convergent region is

$$\Xi_{[134]} = \{(y_1, y_2, y_3) \Big| 1 < |y_3|, |y_1| < |y_3|, |y_2| < |y_3|\} , \tag{2.31}$$

which shows that $\Phi^{(1)}_{[134]}(\alpha, z)$ is in neighborhood of regular singularity ∞.

II. 3-loop vacuum on compact manifold

- In a similar way, we can obtain other seven hypergeometric solutions which are consistent with the basis of integer lattice $\mathbf{B}_{134}$, and the convergent region is also $\Xi_{[134]}$.
- The above eight hypergeometric series solutions $\Phi^{(i)}_{[134]}(\alpha, z)$ whose convergent region is $\Xi_{[134]}$ can constitute a fundamental solution system.
- Multiplying one of the row vectors of the matrix $\mathbf{B}_{134}$ by -1, the induced integer matrix can also be chosen as a basis of the integer lattice space of certain hypergeometric series.

II. 3-loop vacuum on compact manifold

- Taking 3 row vectors of the following matrix as the basis of integer lattice,

$$\mathbf{B}_{\tilde{1}34} = \mathrm{diag}(-1,1,1)\cdot\mathbf{B}_{134}$$
$$= \begin{pmatrix} -1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \end{pmatrix}, \quad (2.32)$$

one obtains eight hypergeometric series solutions $\Phi^{(i)}_{[\tilde{1}34]}(\alpha, z)$ $(i = 1, \cdots, 8)$ similarly. The convergent region is

$$\Xi_{[\tilde{1}34]} = \{(y_1,\ y_2,\ y_3)\Big|\,|y_1| < 1,\ |y_2| < 1,\ |y_3| < 1\}\,, \quad (2.33)$$

which shows that $\Phi^{(i)}_{[\tilde{1}34]}(\alpha, z)$ are in neighborhood of regular singularity 0 and can constitute a fundamental solution system.

II. 3-loop vacuum on compact manifold

- Taking 3 row vectors of the following matrix as the basis of integer lattice,

$$\mathbf{B}_{1\tilde{3}4} = \mathrm{diag}(1,-1,1)\cdot\mathbf{B}_{134} = \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \end{pmatrix}, \quad (2.34)$$

one obtains eight hypergeometric series solutions $\Phi^{(i)}_{[1\tilde{3}4]}(\alpha, z)$ $(i = 1, \cdots, 8)$ similarly. The convergent region is

$$\Xi_{[1\tilde{3}4]} = \{(y_1,\ y_2,\ y_3)\Big|1 < |y_1|,\ |y_2| < |y_1|,\ |y_3| < |y_1|\},\quad (2.35)$$

which shows that $\Phi^{(i)}_{[1\tilde{3}4]}(\alpha, z)$ are in neighborhood of regular singularity ∞ and can constitute a fundamental solution system.

II. 3-loop vacuum on compact manifold

- Taking 3 row vectors of the following matrix as the basis of integer lattice,

$$\mathbf{B}_{13\tilde{4}} = \mathrm{diag}(1,1,-1)\cdot\mathbf{B}_{134} = \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}, \quad (2.36)$$

one obtains eight hypergeometric series solutions $\Phi^{(i)}_{[13\tilde{4}]}(\alpha, z)\ (i = 1, \cdots, 8)$ similarly. The convergent region is

$$\Xi_{[1\tilde{3}4]} = \{(y_1,\ y_2,\ y_3)\Big| 1 < |y_2|,\ |y_1| < |y_2|,\ |y_3| < |y_2|\}, \quad (2.37)$$

which shows that $\Phi^{(i)}_{[13\tilde{4}]}(\alpha, z)$ are in neighborhood of regular singularity ∞ and can constitute a fundamental solution system.

III. 3-loop vacuum on compact manifold

- i9-8th, 32GB:

```
SumLim = 15;
ParameterSub = {Dϵ → 4 - 2 × 0.001, ϵ → 0.001, a1 → 1,
   a2 → 1, a3 → 1, a4 → 1, m4 → 0.01, m1 → 0.02, m2 → 10 , m3 → 0.04 };
NumericalSum[SeriesSolution, ParameterSub, SumLim];
```

Numerical result = -7.5628×10^{8}

Time Taken 1.55064 seconds

```
FIESTAEvaluate[MomentumRep, LoopMomenta, InvariantList, ParameterSub];
```

FIESTA Value = -7.56285×10^{8}

Time Taken 160.037 seconds

- i9-13th, 64GB:

```
SumLim = 15;
ParameterSub = {Dϵ → 4 - 2 × 0.001, ϵ → 0.001, a1 → 1,
   a2 → 1, a3 → 1, a4 → 1, m4 → 0.01, m1 → 0.02, m2 → 10 , m3 → 0.04};
NumericalSum[SeriesSolution, ParameterSub, SumLim];
```

Numerical result = -7.5628×10^{8}

Time Taken 1.16336 seconds

```
FIESTAEvaluate[MomentumRep, LoopMomenta, InvariantList, ParameterSub];
```

FIESTA Value = -7.56285×10^{8}

Time Taken 53.2344 seconds

III. 1-loop self-energy on Grassmannian

1. $m_1^2 = m_2^2 = 0$

- Adopting Feynman parametric representation, we get the integral of zero virtual masses as

$$A_{1SE}(p^2, 0, 0) = \left(\Lambda^2_{\rm RE}\right)^{2-D/2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2(q+p)^2}$$

$$= \frac{i\Gamma(2-\frac{D}{2})\left(\Lambda^2_{\rm RE}\right)^{2-D/2}}{(4\pi)^{D/2}} \int_0^1 dt_1 t_1^{D/2-2}(p^2 t_1 - p^2)^{D/2-2}$$

$$= \frac{i\Gamma(2-\frac{D}{2})\left(\Lambda^2_{\rm RE}\right)^{2-D/2}}{(4\pi)^{D/2}} \int_0^1 \omega_2(t) t_1^{D/2-2} t_2^{2-D}(p^2 t_1 + p^2 t_2)^{D/2-2} \text{ ,(3.1)}$$

with the homogeneous coordinate $t_2 = -1$, the volume element of projective line $\omega_2(t) = t_2 dt_1 - t_1 dt_2$.

- Feng, Zhang, Chang, Feynman integrals of Grassmannians, *PRD* 106(2022)116025 [arXiv: 2206.04224].

III. 1-loop self-energy

1. $m_1^2 = m_2^2 = 0$

- The integral

$$A_{1SE}(p^2,0,0) \propto \int_0^1 \omega_2(t) t_1^{D/2-2} t_2^{2-D} (p^2 t_1 + p^2 t_2)^{D/2-2} \,, \qquad (3.2)$$

can be embedded in the subvariety of the Grassmannian $G_{2,3}$, with splitting local coordinates as

$$A^{1SE} = \begin{pmatrix} 1 & 0 & p^2 \\ 0 & 1 & p^2 \end{pmatrix} . \qquad (3.3)$$

- first row: integration variable t_1, second row: t_2,
first column: power function $t_1^{D/2-2}$, second column: t_2^{2-D},
third column: $(z_{1,3}t_1 + z_{2,3}t_2)^{D/2-2} = (t_1 p^2 + t_2 p^2)^{D/2-2}$.

III. 1-loop self-energy on Grassmannian

1. $m_1^2 = m_2^2 = 0$

$$\text{Splitting local coordinates: } A^{1SE} = \begin{pmatrix} 1 & 0 & p^2 \\ 0 & 1 & p^2 \end{pmatrix} .$$

$$A_{1SE}(p^2, 0, 0) \propto \int_0^1 \omega_2(t) t_1^{D/2-2} t_2^{2-D} (p^2 t_1 + p^2 t_2)^{D/2-2} ,$$

satisfies the following GKZ-system

$$\begin{aligned} &\Big\{\vartheta_{1,1} + \vartheta_{1,3}\Big\} A_{1SE} = -A_{1SE} , \quad \Big\{\vartheta_{2,2} + \vartheta_{2,3}\Big\} A_{1SE} = -A_{1SE} , \\ &\vartheta_{1,1} A_{1SE} = (\frac{D}{2} - 2) A_{1SE} , \quad \vartheta_{2,2} A_{1SE} = (2 - D) A_{1SE} , \\ &\Big\{\vartheta_{1,3} + \vartheta_{2,3}\Big\} A_{1SE} = (\frac{D}{2} - 2) A_{1SE} , \end{aligned} \tag{3.4}$$

where the Euler operator $\vartheta_{i,j} = z_{i,j} \partial / \partial z_{i,j}$.

$$\text{Exponent matrix: } \begin{pmatrix} \frac{D}{2} - 2 & 0 & 1 - \frac{D}{2} \\ 0 & 2 - D & D - 3 \end{pmatrix} . \tag{3.5}$$

III. 1-loop self-energy on Grassmannian

1. $m_1^2 = m_2^2 = 0$

- Exponent matrix:

$$\begin{pmatrix} \frac{D}{2}-2 & 0 & 1-\frac{D}{2} \\ 0 & 2-D & D-3 \end{pmatrix}.$$

- $G_{2,3}$ splitting local coordinates:

$$A^{1SE} = \begin{pmatrix} 1 & 0 & p^2 \\ 0 & 1 & p^2 \end{pmatrix}.$$

- One obtains solution of the GKZ-system

$$A_{1SE}(p^2,0,0) = C_{1SE}^{(0)}\left(p^2\right)^{1-D/2}\left(p^2\right)^{D-3} = C_{1SE}^{(0)}\left(p^2\right)^{D/2-2}. \tag{3.6}$$

III. 1-loop self-energy on Grassmannian

2. $m_1^2 = 0$, $m_2^2 \neq 0$

- Adopting Feynman parametric representation

$$A_{1SE}(p^2, 0, m_2^2) = \left(\Lambda_{RE}^2\right)^{2-D/2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2((q+p)^2 - m_2^2)}$$

$$= \frac{i\Gamma(2-\frac{D}{2})\left(\Lambda_{RE}^2\right)^{2-D/2}}{(-)^{2-D/2}(4\pi)^{D/2}} \int_0^1 dt_1 dt_3 \frac{t_1^{D/2-2}\delta(t_1 + t_3 - 1)}{(t_3 p^2 - m_2^2)^{2-D/2}}$$

$$= \frac{i\Gamma(2-\frac{D}{2})\left(\Lambda_{RE}^2\right)^{2-D/2}}{(-)^{2-D/2}(4\pi)^{D/2}} \int \omega_3(t) \frac{t_1^{D/2-2} t_2^{2-D}\delta(t_1 + t_2 + t_3)}{(t_3 p^2 + t_2 m_2^2)^{2-D/2}} \quad (3.7)$$

with homogeneous coordinate $t_2 = -1$, volume element of projective space $\omega_3(t) = t_1 dt_2 dt_3 - t_2 dt_1 dt_3 + t_3 dt_1 dt_2$.

III. 1-loop self-energy on Grassmannian

2. $m_1^2 = 0$**,** $m_2^2 \neq 0$

- The integral

$$A_{1SE}(p^2, 0, m_2^2) \propto \int \omega_3(t) \frac{t_1^{D/2-2} t_2^{2-D} \delta(t_1 + t_2 + t_3)}{(t_3 p^2 + t_2 m_2^2)^{2-D/2}} \,,$$

can be embedded in the subvariety of the Grassmannian $G_{3,5}$, with splitting local coordinates as

$$A^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & m_2^2 \\ 0 & 0 & 1 & 1 & p^2 \end{pmatrix} . \qquad (3.8)$$

- first row: t_1, second row: t_2, third row: t_3, first column: $t_1^{D/2-2}$, second column: t_2^{2-D}, third column: $t_3^0 = 1$, fourth column represents the function $\delta(t_1 + t_2 + t_3)$, fifth column: $(z_{1,5} t_1 + z_{2,5} t_2 + z_{3,5} t_3)^{D/2-2} = (t_2 m_2^2 + t_3 p^2)^{D/2-2}$.

III. 1-loop self-energy on Grassmannian

2. $m_1^2 = 0$, $m_2^2 \neq 0$

$$A_{1SE}(p^2, 0, m_2^2) \propto \int \omega_3(t) \frac{t_1^{D/2-2} t_2^{2-D} \delta(t_1 + t_2 + t_3)}{(t_3 p^2 + t_2 m_2^2)^{2-D/2}} ,$$

satisfies the following GKZ-system

$$\Big\{\vartheta_{1,1} + \vartheta_{1,4}\Big\} A_{1SE} = -A_{1SE} , \quad \Big\{\vartheta_{2,2} + \vartheta_{2,4} + \vartheta_{2,5}\Big\} A_{1SE} = -A_{1SE} , \tag{3.9}$$
$$\Big\{\vartheta_{3,3} + \vartheta_{3,4} + \vartheta_{3,5}\Big\} A_{1SE} = -A_{1SE} , \quad \vartheta_{1,1} A_{1SE} = (\frac{D}{2} - 2) A_{1SE} , \quad \vartheta_{2,2} A_{1SE} = (2-D) A_{1SE} ,$$
$$\vartheta_{3,3} A_{1SE} = 0 , \quad \Big\{\vartheta_{1,4} + \vartheta_{2,4} + \vartheta_{3,4}\Big\} A_{1SE} = -A_{1SE} , \quad \Big\{\vartheta_{2,5} + \vartheta_{3,5}\Big\} A_{1SE} = (\frac{D}{2} - 2) A_{1SE} .$$

Exponent matrix:

$$\left(\begin{array}{ccccc} \frac{D}{2} - 2 & 0 & 0 & 1 - \frac{D}{2} & 0 \\ 0 & 2 - D & 0 & \alpha_{2,4} & \alpha_{2,5} \\ 0 & 0 & 0 & \alpha_{3,4} & \alpha_{3,5} \end{array} \right) , \tag{3.10}$$

$$\alpha_{2,4} + \alpha_{2,5} = D - 3, \ \alpha_{3,4} + \alpha_{3,5} = -1, \ \alpha_{2,4} + \alpha_{3,4} = \frac{D}{2} - 2, \ \alpha_{2,5} + \alpha_{3,5} = \frac{D}{2} - 2.$$

III. 1-loop self-energy on Grassmannian

2. $m_1^2 = 0$**,** $m_2^2 \neq 0$

- Dual space of the GKZ-system: 3×5 matrix $(0_{3\times 3} | E_3^{(1)})$ with

$$E_3^{(1)} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} . \tag{3.11}$$

- Integer lattice $(0_{3\times 3} | nE_3^{(1)})$ $(n \geq 0)$ is compatible with two choices of the exponents. Through

$$\alpha_{2,4} + \alpha_{2,5} = D - 3,\ \alpha_{3,4} + \alpha_{3,5} = -1,\ \alpha_{2,4} + \alpha_{3,4} = \frac{D}{2} - 2,\ \alpha_{2,5} + \alpha_{3,5} = \frac{D}{2} - 2,$$

the first choice is written as

$$\alpha_{2,4} = 0,\ \alpha_{2,5} = D - 3,\ \alpha_{3,4} = \frac{D}{2} - 2,\ \alpha_{3,5} = 1 - \frac{D}{2} . \tag{3.12}$$

III. 1-loop self-energy on Grassmannian

2. $m_1^2 = 0$, $m_2^2 \neq 0$

$$\text{Splitting local coordinates: } A^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & m_2^2 \\ 0 & 0 & 1 & 1 & p^2 \end{pmatrix}.$$

$$\text{Integer lattice } (0_{3\times 3}\big| nE_3^{(1)}): \quad nE_3^{(1)} = \begin{pmatrix} 0 & 0 \\ n & -n \\ -n & n \end{pmatrix}.$$

$$\text{Exponent matrix: } \begin{pmatrix} \frac{D}{2}-2 & 0 & 0 & 1-\frac{D}{2} & 0 \\ 0 & 2-D & 0 & 0 & D-3 \\ 0 & 0 & 0 & \frac{D}{2}-2 & 1-\frac{D}{2} \end{pmatrix}.$$

Hypergeometric function:

$$\psi^{(1)}_{\{1,2,3\}} \sim (m_2^2)^{\alpha_{2,5}}(p^2)^{\alpha_{3,5}} \sum_{n=0}^{\infty} \frac{\Gamma(-\alpha_{2,5}+n)\Gamma(-\alpha_{3,4}+n)}{\Gamma(1+\alpha_{2,4}+n)\Gamma(1+\alpha_{3,5}+n)} (m_2^2)^{-n}(p^2)^n$$

$$\sim (m_2^2)^{D-3}(p^2)^{1-D/2} \sum_{n=0}^{\infty} \frac{\Gamma(3-D+n)}{n!} \left(\frac{p^2}{m_2^2}\right)^n . \qquad (3.13)$$

III. 1-loop self-energy on Grassmannian

2. $m_1^2 = 0$**,** $m_2^2 \neq 0$

- For integer lattice $(0_{3\times 3} | nE_3^{(1)})$ $(n \geq 0)$, second choice is written as

$$\alpha_{3,5} = 0,\ \alpha_{2,4} = \frac{D}{2} - 1,\ \alpha_{2,5} = \frac{D}{2} - 2,\ \alpha_{3,4} = -1\ . \quad (3.14)$$

- Adopting integer lattice and the corresponding exponents matrices, we obtain hypergeometric function as

$$\psi^{(2)}_{\{1,2,3\}}(p^2,\ 0,\ m_2^2)$$
$$\sim (m_2^2)^{\alpha_{2,5}}(p^2)^{\alpha_{3,5}} \sum_{n=0}^{\infty} \frac{\Gamma(-\alpha_{2,5}+n)\Gamma(-\alpha_{3,4}+n)}{\Gamma(1+\alpha_{2,4}+n)\Gamma(1+\alpha_{3,5}+n)} (m_2^2)^{-n}(p^2)^n$$
$$\sim (m_2^2)^{D/2-2} \sum_{n=0}^{\infty} \frac{\Gamma(2-\frac{D}{2}+n)}{\Gamma(\frac{D}{2}+n)} \left(\frac{p^2}{m_2^2}\right)^n . \quad (3.15)$$

III. 1-loop self-energy on Grassmannian

2. $m_1^2 = 0$, $m_2^2 \neq 0$

- For integer lattice $(0_{3\times 3} \Big| -nE_3^{(1)})$ $(n \geq 0)$, two possibilities:

$$\alpha_{2,5} = 0,\ \alpha_{2,4} = D-3,\ \alpha_{3,4} = 1-\frac{D}{2},\ \alpha_{3,5} = \frac{D}{2}-2\ ; \quad (3.16)$$

$$\alpha_{3,4} = 0,\ \alpha_{2,4} = \frac{D}{2}-2,\ \alpha_{2,5} = \frac{D}{2}-1,\ \alpha_{3,5} = -1\ . \quad (3.17)$$

- Adopting integer lattice and exponents matrices, we obtain two linear independent hypergeometric functions as

$$\psi^{(3)}_{\{1,2,3\}}(p^2,\ 0,\ m_2^2) \sim (p^2)^{D/2-2} \sum_{n=0}^{\infty} \frac{\Gamma(3-D+n)}{n!} \left(\frac{m_2^2}{p^2}\right)^n\ ; \quad (3.18)$$

$$\psi^{(4)}_{\{1,2,3\}}(p^2,\ 0,\ m_2^2) \sim \frac{(m_2^2)^{D/2-1}}{p^2} \sum_{n=0}^{\infty} \frac{\Gamma(2-\frac{D}{2}+n)}{\Gamma(\frac{D}{2}+n)} \left(\frac{m_2^2}{p^2}\right)^n . \quad (3.19)$$

III. 1-loop self-energy on Grassmannian

2. $m_1^2 = 0$, $m_2^2 \neq 0$

- $\det(A^{(1)}_{\{1,2,3\}}) = 1$, $G_{3,5}$ splitting local coordinates

$$A^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & m_2^2 \\ 0 & 0 & 1 & 1 & p^2 \end{pmatrix} . \qquad (3.20)$$

Convergent regions: $|1/x| \leq 1$, $|x| \leq 1$ $(x = m_2^2/p^2)$. Neighborhoods: $x = \infty,\ 0$.

- $\det(A^{(1)}_{\{1,2,5\}}) = p^2$, $G_{3,5}$ splitting local coordinates

$$\left(A^{(1)}_{\{1,2,5\}}\right)^{-1} \cdot A^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -\frac{m_2^2}{p^2} & 1 - \frac{m_2^2}{p^2} & 0 \\ 0 & 0 & \frac{1}{p^2} & \frac{1}{p^2} & 1 \end{pmatrix} . \qquad (3.21)$$

Convergent regions: $|1/(1-1/x)| \leq 1$, $|1-1/x| \leq 1$. Neighborhoods: $x = 0,\ 1$.

- $\det(A^{(1)}_{\{1,3,5\}}) = -m_2^2$, $\det(A^{(1)}_{\{2,3,4\}}) = 1$, $\det(A^{(1)}_{\{2,4,5\}}) = -p^2$, $\det(A^{(1)}_{\{3,4,5\}}) = m_2^2$.
- 24 fundamental solutions

III. 1-loop self-energy on Grassmannian

3. $m_1^2 \neq 0$**,** $m_2^2 \neq 0$

- Adopting Feynman parametric representation

$$A_{1SE}(p^2, m_1^2, m_2^2) = \left(\Lambda^2_{\rm RE}\right)^{2-D/2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 - m_1^2)((q+p)^2 - m_2^2)}$$

$$\propto \int_0^1 dt_1 dt_2 \delta(t_1 + t_2 - 1)(t_1 t_2 p^2 - t_1 m_1^2 - t_2 m_2^2)^{D/2-2}$$

$$\propto \int \omega_3(t)\delta(t_1 + t_2 + t_3) t_3^{2-D}(t_1 t_2 p^2 + t_1 t_3 m_1^2 + t_2 t_3 m_2^2)^{D/2-2} \,. \quad (3.22)$$

- The integral can be embedded in the subvariety of the Grassmannian $G_{3,6}$, with splitting local coordinates as

$$A^{(1S)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & p^2 \\ 0 & 1 & 0 & 1 & 1 & m_2^2 \\ 0 & 0 & 1 & 1 & 1 & m_1^2 \end{pmatrix} . \quad (3.23)$$

- 72 fundamental solutions

IV. Summary

- Using Mellin-Barnes representation and Miller's transformation, we derive GKZ hypergeometric systems of Feynman integrals. In the neighborhoods of origin 0 including infinity ∞, we can obtain analytical hypergeometric series solutions through GKZ-systems.
- Feynman integrals also can be taken as functions on the subvarieties of Grassmannians through homogenizing the parametric representation. The GKZ-systems can be obtained in splitting local coordinates. Fundamental solution systems in neighborhoods of all regular singularities, using matrices of integer lattice and exponent matrices.
- multi-loop diagrams: the sizes of Grassmannians are too large to construct the fundamental systems through Feynman parametric representations. To efficiently derive the solution, we can embed the integrals into the subvarieties of Grassmannians using α-parametric representation.

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