Local spacetime effects of dark energy

— Glasses of God as a gift

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Current detections for dark energy

Einstein equation: dark energy $= \Lambda$? (Weinberg et al. 2013)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} + g_{\mu\nu} \Lambda$$

So far the existing evidences of dark energy are mostly indirect.

$\Downarrow$

(1) The GODDESS mission proposed by NASA: (Nan et al. 2018)
$\rightarrow$ isolate the new force field signal from overwhelmingly stronger gravity effects and achieve a direct detection of dark energy as a scalar field

(2) The electron recoil excess reported by the XENON1T:
$\rightarrow$ a non-gravitational signature of dark energy ? (Vagnozzi et al. 2021)

Dark energy and its local spacetime effects

- Dark energy is characterized by an EoS parameter w, which may evolve with the cosmological redshift z, namely $w = w(z)$.
- However, w can be treated as a constant on astrophysical scales (He & Zhang 2017), like our Solar System.
- The local repulsion from dark energy can be described well by **the SdS$_w$ metric** (He & Zhang 2017, Zhang[2] 2023), exactly as follows:

$$\mathrm{d}S_w^2 = -\left[1-2\frac{M}{r}-2\left(\frac{r_o}{r}\right)^{3w+1}\right]\mathrm{d}t^2 + \frac{1}{\left[1-2\frac{M}{r}-2\left(\frac{r_o}{r}\right)^{3w+1}\right]}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right)$$

The w-term

w: the equation-of-state parameter of dark energy

$$w = \sum_i p_i \Big/ \sum_i \rho_i < -\frac{1}{3}$$

- p_i: the pressure. ρ_i: the energy density.
- Each i represents the ith contribution described by a different dark energy model.
- There is only one w-term responsible for all these dark energy contributions in the SdS_w metric.
- When $w = -1$, the metric reduces to the well-known Schwarzschild-de Sitter (SdS) one.

A long-term debate

- In the early 1980s, Islam showed that the light orbital equation is independent of Λ in the SdS spacetime. Since then it was generally believed that dark energy plays no role on the bending of light.
- By 2007, Rindler and Ishak had concluded that Λ contributes to the deflection of light after considering the measurements done by observers
- However, the conclusion has led to a **debate*** of over 10 years.

  ***The difficulty of understanding the role of dark energy in light bending: the SdS spacetime is not flat at spatial infinity.**

- Ten years later in 2017, He and Zhang extended the debate to the general SdS_w spacetime.

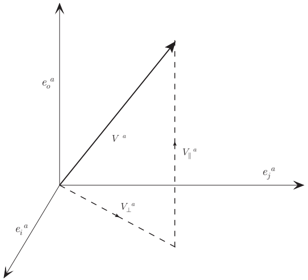
Projection tensors

For any observer, let U^a be their four-velocity with $U^a U_a = -1$. Let

$$h_{ab} = g_{ab} + U_a U_b \text{ and } \pi_{ab} = -U_a U_b.$$

$$\Downarrow$$

$$h_{ab} h^b{}_c = h_{ac}, \quad h_{ab} \pi^b{}_c = 0, \quad \pi_{ab} \pi^b{}_c = \pi_{ac}$$



Indeed, h_{ab} and π_{ab} are space-like and time-like projection tensors, respectively.

$$h^a{}_b U^b = 0, \quad h_{ab} U^b = 0,$$
$$\pi^a{}_b U^b = U^a, \quad \pi_{ab} U^b = U_a.$$

Clearly, $h^a{}_b$ projects the four-vector onto the local space of the observer.

$$V_\perp{}^a = h^a{}_b V^b, \quad V_\parallel{}^a = \pi^a{}_b V^b, \quad V_{\perp a} = h_{ab} V^b, \quad V_{\parallel a} = \pi_{ab} V^b$$

Proper acceleration

- In theories of stationary spacetime, for any observer, their four-acceleration is usually nonzero and given by (Wald Robert M 1984)

$$\hat{A}^a = U^b \nabla_b U^a$$

- Let V^a be the four-velocity of a test-particle moving on a geodesic and passing through the local space of the observer, then the observed three-acceleration by this observer would be

$$\hat{a}^a = -\hat{A}^a + \frac{2}{(V^c U_c)^2} (V^b \hat{A}_b) V_\perp{}^a \tag{1}$$

- Setting $V^a = U^a \rightarrow \hat{a}^a = -\hat{A}^a$. In this case, $\hat{a}^a$ is measured in the rest frame of the test-particle, named as the *proper acceleration*.
- If the instantaneous observer is static relative to the gravitational field, the measured proper acceleration $\hat{a}^a$ by this observer is just the *gravitational 3-force* $\vec{g} = -\vec{A}$, especially in the Newtonian limit.

Physical separation

- In the case of some small physical separation, δ^a, between the test-particle and observer, we have the proper three-acceleration,

$$\hat{a}^a = -(1 + \delta^b \nabla_b) \hat{A}^a + R^a{}_{bcd} U^b U^c \delta^d \quad (2)$$

- The separation will lead to some additional δ-terms.
- In GR, the measurements have to be made by the instantaneous observer at the point of the test-particle or a sufficiently small region around the test-particle $\rightarrow$ δ^a is very small in an actual measurement, and thus these δ-terms can be ignored directly.
- Alternatively, the proper acceleration can be obtained by making δ-corrections to the measured acceleration by the observer located at a small separation of δ^a to the test-particle.

Dark force

- The SdS$_w$ metric is static → time-like Killing vector $\mathcal{K}^a = (\partial/\partial t)^a$
- Here the range that we are interested in is $1-2\frac{M}{r}-2\left(\frac{r_o}{r}\right)^{3w+1}>0$.

Then, $\chi = \sqrt{-g_{00}} = \sqrt{1-2\frac{M}{r}-2\left(\frac{r_o}{r}\right)^{3w+1}}$. By definition, we have

$$\hat{A}^a = \nabla^a \ln\chi \quad \text{(Wald Robert M 1984)}$$

- We therefore obtain the following form of the three-acceleration,

$$\begin{aligned}
\hat{a}^a = -\hat{A}^a &= -g^{ab}\hat{A}_b = -\left(g^{\mu\nu}\partial_\mu{}^a\partial_\nu{}^b\right)\left[\frac{1}{\sqrt{-g_{00}}}\frac{\partial\sqrt{-g_{00}}}{\partial r}(\mathrm{d}r)_b\right] \\
&= -\frac{g^{rr}}{\sqrt{-g_{00}}}\frac{\partial\sqrt{-g_{00}}}{\partial r}\left(\frac{\partial}{\partial r}\right)^a \\
&= -\frac{1}{r}\left[\frac{M}{r}+(3w+1)\left(\frac{r_o}{r}\right)^{3w+1}\right]\left(\frac{\partial}{\partial r}\right)^a
\end{aligned}$$

The fifth force

- $e^a_r = \frac{1}{\sqrt{g_{rr}}}\left(\frac{\partial}{\partial r}\right)^a$ → the **proper form** of the three-acceleration

$$\hat{a}^a = -\frac{1}{r}\frac{1}{\sqrt{1-2\frac{M}{r}-2\left(\frac{r_o}{r}\right)^{3w+1}}}\left[\frac{M}{r}+(3w+1)\left(\frac{r_o}{r}\right)^{3w+1}\right]e^a_r \qquad (3)$$

- Besides the attractive Newtonian force, there exits a **dark force** (generated by dark energy) on the massive test-particle.
- The two forces are on equal footing. The observational data requires $3w+2<0$ at 6σ level (Aubourg et al. 2015). Thus, the w–term leads to a repulsive dark force.
- The proper form of **the total gravitational force** is independent of specific dark energy models.

The fifth force

- By setting $w = -1$, we rewrite it in a more familiar form

$$\vec{g} = -\frac{1}{r}\frac{1}{\sqrt{1-2\frac{M}{r}-2\frac{\Lambda}{6}r^2}}\left(\frac{M}{r}-\frac{\Lambda}{3}r^2\right)\hat{\boldsymbol{r}},$$

with $|\hat{\boldsymbol{r}}| = 1$. → the cosmological constant problem

- Under the weak field approximation, one gets

$$\vec{g} \simeq -\left(\frac{M}{r^2}-\frac{\Lambda}{3}r\right)\hat{\boldsymbol{r}},$$

which agrees with the dark force shown in Ho & Hsu (2015).

- The dark force is also referred to as '*the fifth force*'. It will lead to the deviation from the inverse square law behavior.

Gravitational field strength

- As seen from the static observer with respect to the gravitational field, the strength of the three-force $\vec{g}$ on the test-particle, also known as **the gravitational field strength**, is:

$$
\begin{aligned}
g &= \sqrt{\hat{a}^a \hat{a}_a} = \sqrt{\hat{A}^a \hat{A}_a} \\
&= \sqrt{\left[\frac{g^{rr}}{\sqrt{-g_{00}}} \frac{\partial \sqrt{-g_{00}}}{\partial r} \left(\frac{\partial}{\partial r}\right)^a\right]\left[\frac{1}{\sqrt{-g_{00}}} \frac{\partial \sqrt{-g_{00}}}{\partial r} (\mathrm{d}r)_a\right]} \\
&= \left|\frac{1}{\sqrt{(-g_{00})\, g_{rr}}} \left(\frac{\partial \sqrt{-g_{00}}}{\partial r}\right)\right| \\
&= \frac{1}{r} \frac{1}{\sqrt{1-2\frac{M}{r}-2\left(\frac{r_o}{r}\right)^{3w+1}}} \left|\frac{M}{r}+(3w+1)\left(\frac{r_o}{r}\right)^{3w+1}\right|.
\end{aligned}
$$

Searching for evidence of the dark force

- When taking $r_o \to \infty$, one has

$$g \simeq \frac{M}{r^2} \frac{1}{\sqrt{1-2\frac{M}{r}}},$$

which is just the **Rindler's form** of the field strength (Rindler 2006).

- In this case, for large r, the field strength recovers the **Newtonian inverse-square law** in the Newtonian limit, namely

$$g \simeq \frac{M}{r^2}.$$

- The GODDESS team is trying to detect any possible **deviation** from the inverse square law behavior (Nan et al. 2018), **although they knew nothing about the specific form of the dark force.**

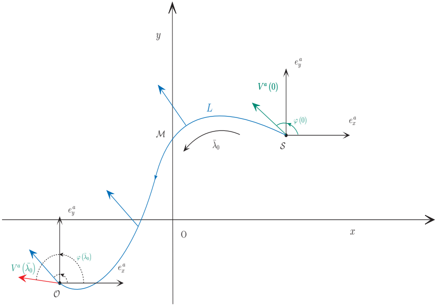
Critical radius r_{cri}

- Requiring $\vec{g} = 0$ or $g = 0$, we derive the critical radius r_{cri} without any further approximations,

$$r_{\text{cri}} = r_o \left(\frac{M}{|3w+1| \, r_o} \right)^{-\frac{1}{3w}} \tag{4}$$

- At $r = r_{\text{cri}}$, the repulsive dark force can balance the attractive Newtonian force.
- The repulsive dark force will dominate over the Newtonian attraction in the outer region with $r \gtrsim r_{\text{cri}}$.
- The GODDESS team is likely to detect the dark force at close distances to the Sun. However, the critical radius tells us that it is wrong, and the space experiment should take place at long distances, like $r \gtrsim 0.1 \; r_{\text{cri}}$.

How to quantify the gravitational deflection



- Denote $V^a = V^a(\lambda)$ as the 4-momentum of any light ray at point λ. Let $A^a = V^a(0)$ and $B^a = V^a(\bar{\lambda}_0)$ be 4-momenta of the incident and outgoing rays, respectively.
- How to define the deflection angle in any curved spacetime ?

$$\text{Parallel transport ?} \quad \cos\angle(A,B) = \frac{A_a B^a}{|A|\,|B|} \leftarrow |A| = 0 \ \& \ |B| = 0 \ ? \quad \dots ?$$

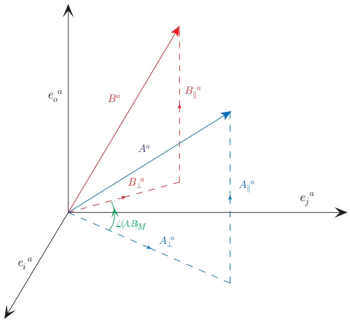
$\Longrightarrow$ W. Rindler, G. W. Gibbons, ...

Measurable intersection angle

Let $h_{ab} = g_{ab} + U_a U_b$ and $\pi_{ab} = -U_a U_b$.

$$\Downarrow$$

$$V_{\perp}{}^{a} = h^{a}{}_{b} V^{b}, \quad V_{\parallel}{}^{a} = \pi^{a}{}_{b} V^{b}, \quad V_{\perp a} = h_{ab} V^{b}, \quad V_{\parallel a} = \pi_{ab} V^{b}$$



Denote A^a and B^a as four-momenta of two intersecting light rays, respectively.

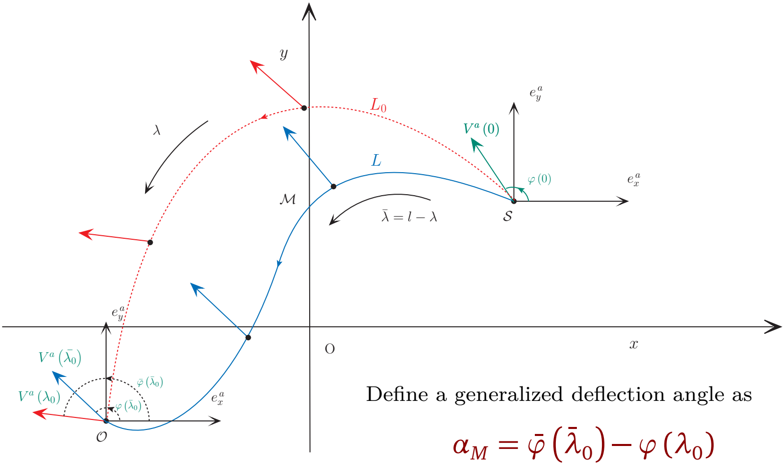
$$\cos\angle(A,B)_M = \frac{A_{\perp a} B_{\perp}{}^{a}}{|A_\perp|\,|B_\perp|}$$

In this way, we define $\angle(A,B)_M$ as the measurable intersection angle.

Physical surface $(\Sigma, \hat{h}_{ab})$

- In theories of gravity, the entire spacetime is uniquely determined by one metric tensor g_{ab}.
- In any **static* spacetime**, there always exists a local space Ξ_p perpendicular to the four-velocity U^a of the static observer at each point p. *: static → extrinsic
- We can paste these local spaces $\{\Xi_p\}$ at different points $\{p\}$ together to get a three-dimensional manifold Ξ, namely $\Xi = \bigcup_p \Xi_p$. It is determined by the metric tensor $h_{ab} = g_{ab}\big|_{\Xi}$.
- In fact, Ξ can be viewed as a physical hypersurface on which measurements may take place.
- The physical surface $(\Sigma, \hat{h}_{ab})$ can be chosen at our convenience.
- For instance, in the SdS_w spacetime, we can construct such a surface by setting $\theta = \frac{\pi}{2}$, without loss of generality.

Gaussian deflection angle



Define a generalized deflection angle as

$$\alpha_M = \bar{\varphi}(\bar{\lambda}_0) - \varphi(\lambda_0)$$

Newly discovered relationships

- By the theorem of uniqueness of parallel transport and the theorem of existence and uniqueness of differential equations, we have

$$\varphi(\lambda_0) - \bar{\varphi}\left(\bar{\lambda}_0\right) = \varphi(l) - \varphi(0)$$

- Parallelly, we also have

$$\bar{\varphi}\left(\bar{\lambda}\right) - \varphi(\lambda) = \bar{\varphi}\left(\bar{\lambda}_0\right) - \varphi(\lambda_0)$$

- These results are all independent of the choice of $V^a(0)$.
- The two relationships hold well for any simple, closed piecewise regular curve γ on Σ.

Geometrization of light bending

- The Gaussian deflection angle is defined as

$$\alpha_M = \bar{\varphi}\left(\bar{\lambda}_0\right) - \varphi\left(\lambda_0\right) \tag{5}$$

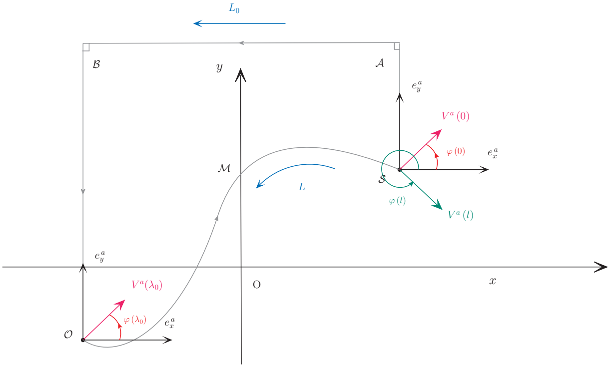
which is applicable to any curved static spacetime.

- By the Gauss-Bonnet theorem, we have $\varphi(l) - \varphi(0) = \iint_D K \mathrm{d}\sigma$
- Combining with the two newly discovered relationships, the Gaussian deflection angle can be further derived as follows,

$$\alpha_M = \bar{\varphi}\left(\bar{\lambda}\right) - \varphi(\lambda) = -\iint_D K \mathbf{d}\sigma \tag{6}$$

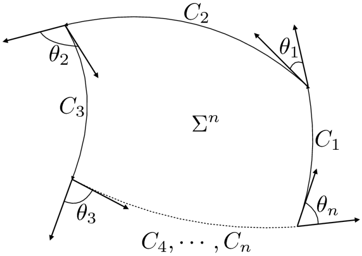
which is independent of specific spacetime models.
⇒ The nature of the bending of light is the curvature of spacetime.

- Now we have geometrized light bending successfully, and extended the definition of the deflection angle to the most general static spacetime.



- Taking the parallel transport of $V^a(0)$ along the path L_0, one has $\varphi(\lambda_0) = \varphi(0)$, due to the *flatness* or the *conformal flatness* of the laboratory area (He & Zhang 2017). $\Longrightarrow$ $\alpha_M = \bar{\varphi}(\bar{\lambda}_0) - \varphi(0)$, which is just the *usual deflection angle*.
- The Gaussian deflection angle α_M is actually a generalized deflection angle. In some special spacetimes, it can reduce to the usual deflection angle (or the weak deflection angle).

Gauss-Bonnet theorem



- Assume that the curve γ is positively oriented, parametrized by arc length λ, and let θ_i and $\gamma(\lambda_i)$ be, respectively, the ith external angle and the ith vertex of γ. Then

$$\sum_i \int_{\lambda_i}^{\lambda_{i+1}} k_g \mathrm{d}\lambda + \iint_D K \mathrm{d}\sigma + \sum_i \theta_i = 2\pi,$$

where $k_g = k_g(\lambda)$ is the geodesic curvature of the regular arcs of γ (do Carmo 2016).

A remarkable relation

The meaning of the Gaussian deflection angle

- The math involved in the *Gaussian deflection angle* is fairly simple.

  (1) Choose D to be a geodesic polygon (that is, polygon whose sides are arcs of geodesics). Denote $\partial\mathrm{D}$ as the closed boundary of the region D, namely $\gamma = \partial\mathrm{D}$.

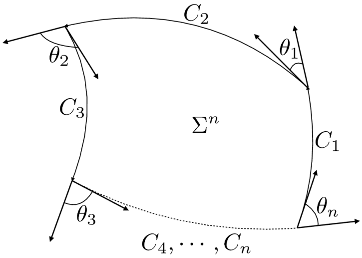
  (2) In the case of gravitational lensing, light follows null geodesics. Thus, the geodesic curvature of $\partial\mathrm{D}$ is zero, namely, $k_g \equiv 0$.

- Then, according to the Gauss-Bonnet theorem, we also have

$$\alpha_M = \sum_i \theta_i - 2\pi \tag{7}$$

$\Longrightarrow$ The ith external angle θ_i of $\partial\mathrm{D}$ is actually the measurable intersection angle by the static observer at the ith vertex $\gamma(\lambda_i)$.

The Meaning of the Gaussian deflection angle



- The Gaussian deflection angle is equal to the excess over 2π of the sum of the external angles of the geodesic polygon, namely

$$\alpha_M = \sum_i \theta_i - 2\pi$$

- Take a geodesic triangle for example. Denote the ith interior angle as ψ_i. Then we have $\alpha_M = \pi - \sum_{i=1}^{3} \psi_i$. If the triangle is on a flat surface, we have $\alpha_M = 0$. Also, on a sphere-like surface, $\alpha_M < 0$. Similarly, on a pseudosphere-like surface, $\alpha_M > 0$.

Performing measurements

- To obtain the Gaussian deflection angle, one needs to measure the external angle θ_i or the interior angle ψ_i at each vertex $\gamma(\lambda_i)$ of $\partial \mathrm{D}$ in advance.
- These external (or interior) angles are actually the intersection angles between two light trajectories (as null geodesics), and each angle can be directly measured by the static observer at each vertex.
- In reality, measurements may be performed by moving observers passing by each vertex. In this case, the measured values by the moving observer for θ_i or ψ_i need to be made relativistic corrections. In fact, the finally measured θ_i or ψ_i by the static observer at each vertex $\gamma(\lambda_i)$ can be obtained by using the general relativistic aberration relationships (Lebedev & Lake 2013).
- In addition, the Gaussian deflection angle, as a global quantity, is fully determined by the integral of the Gaussian curvature K over the lensing patch D, namely the total curvature.
- Using the Gauss-Bonnet theorem, we establish a relation between the global properties of the lensing patch D, like the total curvature, and the local properties of the curve $\partial \mathrm{D}$ composed of null geodesics, such as θ_i or ψ_i at each vertex.
- By definition, K quantifies properties intrinsic to the spacetime surface Σ. Therefore, from this perspective, the Gaussian deflection angle can be used as a potentially interesting probe of the intrinsic properties of spacetime.

Local spacetime effects of dark energy

Come back to the case of the SdS$_w$ spacetime:

- **When $M = 0$, the SdS$_w$ metric is conformally flat.**

  In this case, the light ray travels along a physically straight line, which is the same as the Minkowski vacuum case (He & Zhang 2017). It means that in this case, we can not probe the local spacetime effect of dark energy on the bending of light.

- **For the case of $M \neq 0$, it becomes rather difficult to deal with the SdS$_w$ metric in a similar way.**

  In fact, it is almost impossible to demonstrate what role dark energy plays in the bending of light based on the traditional theories[1] only.

[1] In traditional theories for light deflection, the usual deflection angle plays a central role, and various approaches have been proposed accordingly to probe the bending of light. Hereafter, this kind of approaches are referred to as the traditional approaches.

The SdS$_w$ spacetime

- **In this spacetime, there always exists a physical hypersurface Ξ, locally perpendicular to the four-velocity U^a of the static observer at each point.**

  The subspace Ξ can be globally described by the induced metric tensor,

$$h_{ab} \;\; = \;\; \frac{1}{\left[1-2\frac{M}{r}-2\left(\frac{r_o}{r}\right)^{3w+1}\right]}(\mathrm{d}r)_a(\mathrm{d}r)_b+r^2(\mathrm{d}\theta)_a(\mathrm{d}\theta)_b+r^2\sin^2\theta\,(\mathrm{d}\phi)_a(\mathrm{d}\phi)_b\,.$$

  We then take $\theta=\frac{\pi}{2}$, without lossing generality, and therefore obtain the following metric tensor,

$$\hat{h}_{ab}=\frac{1}{\left[1-2\frac{M}{r}-2\left(\frac{r_o}{r}\right)^{3w+1}\right]}(\mathrm{d}r)_a(\mathrm{d}r)_b+r^2(\mathrm{d}\phi)_a(\mathrm{d}\phi)_b\,.$$

  Thus the subspace Ξ reduces to the (r,ϕ)-plane determined by $\hat{h}_{ab}$.

- **We therefore have constructed a physical surface Σ, characterized by**

$$\mathrm{d}s^2 \;\; = \;\; E\,\mathrm{d}u^2+2F\,\mathrm{d}u\,\mathrm{d}v+G\,\mathrm{d}v^2=\frac{1}{\left[1-2\frac{M}{r}-2\left(\frac{r_o}{r}\right)^{3w+1}\right]}\,\mathrm{d}r^2+r^2\,\mathrm{d}\phi^2.$$

Gaussian curvature K & total curvature K_{tot}

The physical surface is orthogonally parametrized by $(u,v)=(r,\phi)$. In this case,

$$E=\frac{1}{\left[1-2\frac{M}{r}-2\left(\frac{r_o}{r}\right)^{3w+1}\right]},\quad F=0,\quad G=r^2.$$

- **With these, we further derive the Gaussian curvature as**

$$K=-\frac{1}{2\sqrt{EG}}\left(\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v+\left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right)=\frac{\partial}{\partial r}\left(\sqrt{1-2\frac{M}{r}-2\left(\frac{r_o}{r}\right)^{3w+1}}\right)$$

- **The total curvature K_{tot} can be written as**

$$K_{\text{tot}}=\iint_D K\mathrm{d}\sigma=-\frac{1}{2}\oint_{\partial\mathrm{D}}\left(-\frac{(\sqrt{E})_v}{\sqrt{G}}\,\mathrm{d}u+\frac{(\sqrt{G})_u}{\sqrt{E}}\,\mathrm{d}v\right)$$

The role of dark energy in light bending

Finally, the Gaussian deflection angle can be reexpressed as

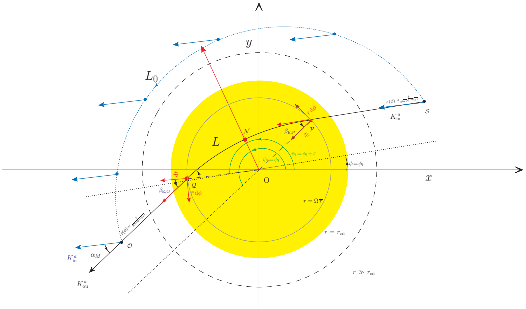
$$\alpha_M = \frac{1}{2}\oint_{\partial \mathrm{D}} \sqrt{1-2\frac{M}{r}-2\left(\frac{r_o}{r}\right)^{3w+1}}\,\mathrm{d}\phi, \tag{8}$$

where $\phi = \phi(r)$ is uniquely determined by the boundary curve $\partial\mathrm{D}$.

- It takes a model-independent form, with various dark energy models described by different w values.
- **Dark energy does contribute light bending via the w-term.**
- It is possible to extract the information about $\left(w, r_o\right)$ via the above formula by making precise measurements for the bending of light.
- Clearly, this will have many applications.

  For instance, as expected from the model predictions of the cosmological constant as the dark energy candidate, we have $w=-1$ and $\Lambda = 6/r_o^2$, which can be tested directly through measuring the bending of light on astrophysical scales, independently of current cosmological observations.

Light orbit equation

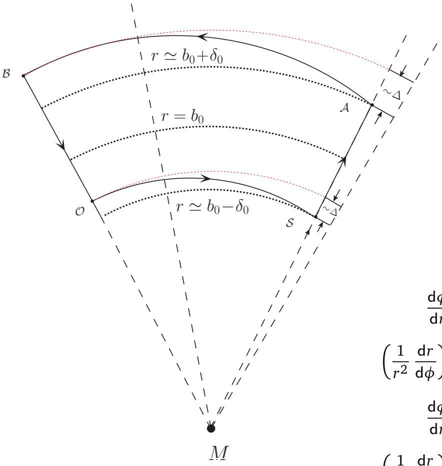


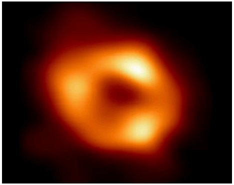
- In the SdS_w spacetime, the LOE is given by (He & Zhang 2017)

$$\left(\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = \frac{1}{b^2} - \frac{1}{r^2}\left[1-2\frac{M}{r}-2\left(\frac{r_o}{r}\right)^{3w+1}\right].$$

- The light orbit is symmetric relative to the straight line $\overline{\mathrm{O}\mathcal{N}}$. At large radius r, it approximately reduces to a physically straight line.

An example of the lensing patch





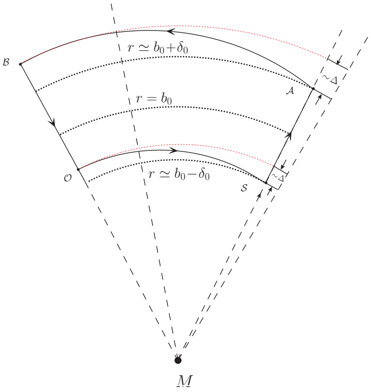
$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = 0 \quad (\overline{\mathcal{S}\mathcal{A}})$$

$$\left(\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = \frac{1}{(b_0+\delta_0)^2} - \frac{1}{r^2}\left[1-2\frac{M}{r}-2\left(\frac{r_o}{r}\right)^{3w+1}\right] \quad (\widetilde{\mathcal{A}\mathcal{B}})$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = 0 \quad (\overline{\mathcal{B}\mathcal{O}})$$

$$\left(\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = \frac{1}{(b_0-\delta_0)^2} - \frac{1}{r^2}\left[1-2\frac{M}{r}-2\left(\frac{r_o}{r}\right)^{3w+1}\right] \quad (\widetilde{\mathcal{O}\mathcal{S}})$$

An order-of-magnitude estimate



- To do this, set $R_{\mathcal{A}} < R_{\mathcal{B}}$ and $R_{\mathcal{S}} < R_{\mathcal{O}}$, exactly as portrayed in the figure, where $R_{\mathcal{P}}$ denotes the radius at which the point $\mathcal{P}$ is located.
- Then use Δ to characterize half the absolute size of the change in the distance to the mass center M from the photon traveling along a given path, such as $\widetilde{\mathcal{AB}}$ or $\widetilde{\mathcal{OS}}$.
- By setting $\delta_0 \ll b_0$, and $\Delta \ll b_0$, we have $r \simeq b_0$ for the closed curve ∂D. Thus the parameter b_0 can be used to characterize the distance of the lensing patch D to the mass center M.
- Now consider the gravitational potential to be fairly weak, $\frac{M}{r} \ll 1$ and $\left(\frac{r_o}{r}\right)^{3w+1} \ll 1$.
- More precisely, further set $\Delta \simeq \frac{R_{\mathcal{B}} - R_{\mathcal{A}}}{2} \sqrt{1 - 2\frac{M}{b_0} - 2\left(\frac{r_o}{b_0}\right)^{3w+1}} \approx \frac{R_{\mathcal{B}} - R_{\mathcal{A}}}{2}$ for the path $\widetilde{\mathcal{AB}}$, and $\Delta \simeq \frac{R_{\mathcal{O}} - R_{\mathcal{S}}}{2} \sqrt{1 - 2\frac{M}{b_0} - 2\left(\frac{r_o}{b_0}\right)^{3w+1}} \approx \frac{R_{\mathcal{O}} - R_{\mathcal{S}}}{2}$ for the path $\widetilde{\mathcal{OS}}$. Let the two paths, $\widetilde{\mathcal{AB}}$ and $\widetilde{\mathcal{OS}}$, have nearly the same Δ. As a result, the lensing patch D may be a long, narrow belt. However, its length can be much smaller than b_0 in our scheme for the direct probe of dark energy.
- Then assume $\left(\frac{r_o}{b_0}\right)^{3w+1} \ll \frac{M}{b_0}$.

An order-of-magnitude estimate

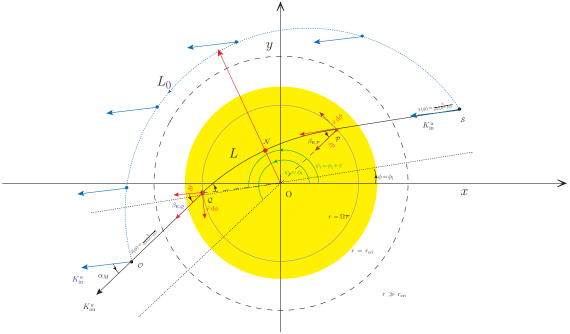
- Under these assumptions and approximations, we can estimate the Gaussian deflection angle for the case of $w \simeq -1$ as follows,

$$\begin{aligned}
\alpha_M &= \frac{1}{2}\oint_{\partial \mathrm{D}} \frac{\sqrt{1-2\frac{M}{r}-2\left(\frac{r_o}{r}\right)^{3w+1}}}{\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)} \mathrm{d}r \\
&= [\ \dots\ ]_{\widetilde{\mathcal{S}\mathcal{A}}} + [\ \dots\ ]_{\widetilde{\mathcal{A}\mathcal{B}}} + [\ \dots\ ]_{\widetilde{\mathcal{B}\mathcal{O}}} + [\ \dots\ ]_{\widetilde{\mathcal{O}\mathcal{S}}} \\
&\sim 0 + \frac{\frac{\Delta}{b_0+\delta_0}}{\sqrt{\frac{M}{b_0+\delta_0}+\left(\frac{b_0+\delta_0}{r_o}\right)^2}} + 0 - \frac{\frac{\Delta}{b_0-\delta_0}}{\sqrt{\frac{M}{b_0-\delta_0}+\left(\frac{b_0-\delta_0}{r_o}\right)^2}} \\
&\sim 0 + \frac{\Delta}{b_0}\sqrt{\frac{b_0}{M}}\left[1-\frac{1}{2}\frac{b_0}{M}\left(\frac{b_0}{r_o}\right)^2\right]\left(1-\frac{1}{2}\frac{\delta_0}{b_0}\right) \\
&\quad +0 - \frac{\Delta}{b_0}\sqrt{\frac{b_0}{M}}\left[1-\frac{1}{2}\frac{b_0}{M}\left(\frac{b_0}{r_o}\right)^2\right]\left(1+\frac{1}{2}\frac{\delta_0}{b_0}\right) \\
&= \left(\frac{\Delta}{b_0}\right)\left(\frac{\delta_0}{b_0}\right)\sqrt{\frac{b_0}{M}}\left[1-\frac{1}{2}\frac{b_0}{M}\left(\frac{b_0}{r_o}\right)^2\right],
\end{aligned}$$

where $[\ \dots\ ]_{\text{part}}$ represents the line integral along a part of the boundary curve $\partial \mathrm{D}$.

- The last line gives an approximate formula quite roughly, but this will suffice for an order-of-magnitude estimate.
- The Gaussian deflection angle takes a simple form, providing a straightforward way to understand the role of dark energy in the bending of light.

Usual deflection angle



- For comparison, calculate the *usual deflection angle* by integrating the LOE (a traditional approach) whereas using new techniques.
- For the cosmological constant case, the usual deflection angle can be approximated as

$$\alpha_M \sim \frac{4M}{b}\left[1+\left(\frac{b}{r_o}\right)^2\right] = \frac{4M}{b} + \frac{2M\Lambda}{3} b,$$

where the Λ term is consistent with the result presented in the literature. When $\Lambda = 0$, it recovers the conventional result, $\alpha_M = \frac{4M}{b}$.

Enhancement mechanism of dark energy effects

$\Delta\alpha_M$: the dark energy correction

- **to the usual deflection angle:**

$$\frac{\Delta\alpha_M}{\alpha_M} \sim \left(\frac{b}{r_o}\right)^2 \tag{9}$$

→ For a real astrophysical system, effects of Λ on the bending of light are too small to be detected. For example, let b equal the size of our Solar System, like $b = 0.1\ r_{\text{cri}}$. Then we have $\frac{\Delta\alpha_M}{\alpha_M} \sim 10^{-18}$.

- **to the Gaussian deflection angle:**

$$\frac{\Delta\alpha_M}{\alpha_M} \sim \frac{b_0}{r_S}\left(\frac{b_0}{r_o}\right)^2 \tag{10}$$

- This is about $\sim \frac{b_0}{r_S}$ times larger than that to the usual deflection angle.
- It is worth rephrasing that we can further enhance the local effect of dark energy via the proper choice of the boundary curve $\partial\mathrm{D}$.

Probing dark energy directly in our Solar System

- In our Solar System, we have $M \sim 1.0\, M_\odot$, where $M_\odot$ is the mass of the Sun.
- Set $b_0 = \mathbf{0.1}\ r_{\text{cri}}$ and $\delta_0 \sim \Delta \sim 1.0\ \mathbf{AU}$ (astronomical unit).

  → In this case, it can be verified that all the assumptions and approximations hold well throughout the derivations of the above formulae.
- Then, we have $\frac{b_0}{r_S} \sim 2.2 \times 10^{14}$.

  → The effect of dark energy on light bending can be enhanced by 14 orders of magnitude.
- We further obtain $\alpha_M \sim 0.63''$ and $\frac{\Delta\alpha_M}{\alpha_M} \sim 2.5 \times 10^{-4}$.

  → We can directly probe the existence of dark energy and measure the EoS parameter w on a much shorter length scale than the size of the Solar-System at $r \sim 0.1\ r_{\text{cri}}$ once a spatial resolution of $\sim 10^{-5}$ $(\times 10^{-3}\uparrow)$ arcseconds can be reached by the current lensing experiments. ⇐ A space mission concept proposed by the Aerospace Corporations: lensing experiments @ ∼ 548 − 900 AU, with a duration of decades (arXiv: 2207.03005).
- In fact, this spatial resolution has been achieved by GRAVITY (for a quasar @ z=0.16), often used to detect the gravitational microlensing events (Sturm et al. 2018).

Summary

1. We geometrized the bending of light using Gauss-Bonnet theorem, and thus stressed a long-term problem in mathematical physics: *Does dark energy affect light bending, and if so, how does it do so?*
2. We suggested a method to overcome the difficulty of measuring the local dark energy effect.
3. Exactly, we have shown how the generalized deflection angle, named as the Gaussian deflection angle, takes a simple form,

$$\alpha_M = \bar{\varphi}\left(\bar{\lambda}\right) - \varphi\left(\lambda\right) = -\iint_D K \, \mathrm{d}\sigma$$

providing a clear way to understand the contribution of dark energy to the bending of light which can be enhanced by 14 orders of magnitude.

One reviewer commented:

"The manuscript definitely renders a new direction of understanding and detecting dark energy directly."

同行评价

同行评价

- Dirac 奖章获得者 Stephen L. Adler 重点引用：

Stephen L. Adler： ABJ 反常的发现者。消除 ABJ 反常是规范理论可重整化的前提，模型构造的基本原则。量子场论教科书，都会有一章专门讲 ABJ 反常。

VII. ACKNOWLEDGEMENT

(Adler 2022)

I wish to thank Zhen Zhang for correspondence in which he called my attention to his paper [4], which stimulated the investigations reported here.

- 剑桥大学的科学家：

(arXiv: 2201.04528)

[3] We also note a recent very interesting proposal for the direct detection of DE on Solar System scales put forward in He & Zhang (2017) and Zhang (2022), exploiting the gravitational deflection of light.

Other closely related problems

- **Gravitational lensing: strong, weak, micro**
- **Formation of structures, like galaxies and clusters of galaxies**
  $\rightarrow$ the dark force will weaken the gravitational interactions, suppress the gravitational clustering process, and slow down the structure formation.
- **The CMB measurement:** $(\Omega_b,\ \Omega_{\rm DM},\ \Omega_\Lambda) \sim (0.04,\ 0.28,\ 0.68)$ **?**
- **Astronomical bounds on neutrino masses:** $\sum m_i \lesssim 0.23$ eV **?**
- **The cosmological constant problem**
- **Propagation of gravitational waves:** $m_{\text{graviton}} =$ **or** $\neq 0$ **?**

  ... ...
- **Applications to searches for physics beyond GR, like the** $f\,(R)$ **gravity** (Buchdahl 1970)**, the Einstein-Gauss-Bonnet gravity** (Glavan & Lin 2020)**, or the higher-order gravity** (Schmidt 1990)

Thanks for your attentions!

Backup slides

Laboratory area

- The SdS_w metric with $M = 0$ is conformally flat.
- For the general case of $M \neq 0$, in the region with $r \gg r_{\mathrm{cri}}$, the Newtonian attraction of matter can be ignored to a great extent, and the gravitational force is dominated by the local repulsion from dark energy.
- When neglecting the Newtonian term completely, we can rewrite the SdS_w metric under coordinate transformations as

$$\mathrm{d}S^2 = \Omega^2(\bar{\mathrm{r}})(-\mathrm{d}\bar{\tau}^2 + \mathrm{d}\bar{\mathrm{r}}^2 + \bar{\mathrm{r}}^2\mathrm{d}\theta^2 + \bar{\mathrm{r}}^2\sin^2\theta\,\mathrm{d}\phi^2).$$

- In this ideal case, dark energy has no measurable effect on the usual light deflection.Hereafter, we name this kind of region as the ***laboratory area***.

Travel of the light ray along a straight line

- Without losing generality, we confine the motion to the $(\bar{\mathrm{r}}, \phi)$-plane of $\theta = \frac{\pi}{2}$. Then the metric reduces to

$$\mathrm{d}S^2 = \Omega^2(\bar{\mathrm{r}})(-\mathrm{d}\bar{\tau}^2 + \mathrm{d}\bar{\mathrm{r}}^2 + \bar{\mathrm{r}}^2\,\mathrm{d}\phi^2).$$

- In any given coordinates η^μ, the energy-momentum 4-vector of a light ray can be defined by $K^a = \frac{\mathrm{d}\eta^\mu}{\mathrm{d}\lambda}\left(\frac{\partial}{\partial \eta^\mu}\right)^a$, where λ is an affine parameter.
- Accordingly, we obtain two Killing vectors: $\xi^a = (\partial/\partial\bar{\tau})^a$ and $\zeta^a = (\partial/\partial\phi)^a$.

→ The metric respects the symmetries of time translation and space rotation.

→ The energy E and the angular momentum L are conserved, respectively.

$$E = -\xi^a K_a = \Omega^2 \frac{\mathrm{d}\bar{\tau}}{\mathrm{d}\lambda} = \text{Constant}, \quad L = \zeta^a K_a = \bar{\mathrm{r}}^2\,\Omega^2 \frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = \text{Constant},$$

where $\Omega = \Omega(\bar{\mathrm{r}})$. Here, E and L are both physical quantities.

Travel of the light ray along a straight line

- Combining with the null condition $\mathrm{d}\bar{\tau}^2 = \mathrm{d}\bar{\mathrm{r}}^2 + \bar{\mathrm{r}}^2\,\mathrm{d}\phi^2$, with $b = \mathrm{L}/\mathrm{E}$, we obtain

$$\left(\frac{1}{\bar{\mathrm{r}}^2}\frac{\mathrm{d}\bar{\mathrm{r}}}{\mathrm{d}\phi}\right)^2 = \frac{1}{b^2} - \frac{1}{\bar{\mathrm{r}}^2}.$$

  Like E and L, the impact parameter b is also a physical quantity.

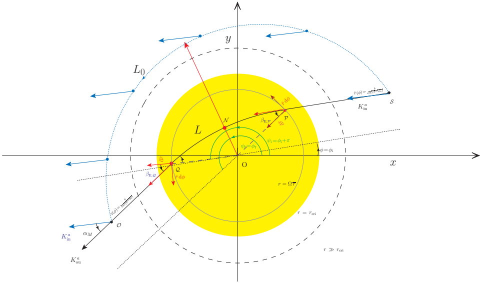
- Furthermore, we rederive this equation as follows,

$$\bar{\mathrm{r}} = \frac{b}{\sin(\phi - \phi_0)},$$

  where ϕ denotes the polar angle measured in a counterclockwise direction from a given *x*-axis (as the polar axis).

- It describes a straight line, and ϕ_0 is the intersection angle between this line and the *x*-axis. → It tells us that the light ray travels along a straight line in any conformally flat space.

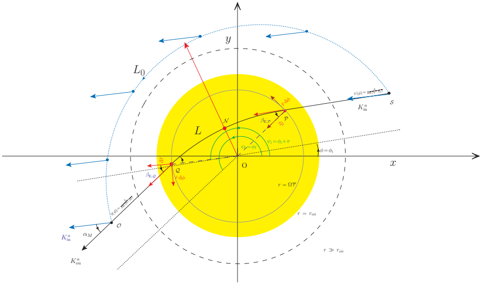
Symmetric light orbit



- In this figure, we illustrate a trajectory of light. $\mathcal{N}$ denotes the point of closest approach of the trajectory, with $(r,\phi)=(r_*,\phi_*)$. At this point, we have $\frac{\mathrm{d}r}{\mathrm{d}\phi}\Big|_{\mathcal{N}}=0$; that is,

$$\frac{1}{b^2}=\frac{1}{r_*^2}\left[1-2\frac{M}{r_*}-2\left(\frac{r_o}{r_*}\right)^{3w+1}\right].$$

- The light orbit intersects with the circle of radius r at two points, namely $\mathcal{P}$ and $\mathcal{Q}$.



- The Euclidean intersection angles $\beta_{E,\mathcal{P}}$ and $\beta_{E,\mathcal{Q}}$ are respectively given by

$$\tan\beta_{\mathrm{E},\mathcal{P}} = \left[\frac{\mathrm{r}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)}{\left(\frac{\mathrm{dr}}{\mathrm{d}\lambda}\right)}\right]_{\mathcal{P}} = \left[\mathrm{r}\left(\frac{\mathrm{d}\phi}{\mathrm{dr}}\right)\right]_{\mathcal{P}}, \qquad \tan\beta_{\mathrm{E},\mathcal{Q}} = \left[\frac{\mathrm{r}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)}{\left(\frac{\mathrm{dr}}{\mathrm{d}\lambda}\right)}\right]_{\mathcal{Q}} = \left[\mathrm{r}\left(\frac{\mathrm{d}\phi}{\mathrm{dr}}\right)\right]_{\mathcal{Q}}.$$

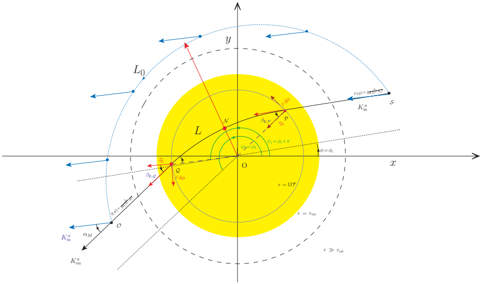
- Since $\mathcal{P}$ and $\mathcal{Q}$ lie at the same radius, we have $\left(\mathrm{r}\,\frac{\mathrm{d}\phi}{\mathrm{dr}}\right)_{\mathcal{P}} = -\left(\mathrm{r}\,\frac{\mathrm{d}\phi}{\mathrm{dr}}\right)_{\mathcal{Q}}$ from the LOE. Thus, one has $\beta_{\mathrm{E},\mathcal{P}} = -\beta_{\mathrm{E},\mathcal{Q}}$. In fact, for any r, we can obtain the same result. In special, at the point of closest approach where $\mathcal{P}$ meets $\mathcal{Q}$, we have $\beta_{\mathrm{E},\mathcal{P}} = -\beta_{\mathrm{E},\mathcal{Q}} = -\pi/2$. It is now clear that $\left|\beta_{\mathrm{E},\mathcal{P}}\right| = \left|\beta_{\mathrm{E},\mathcal{Q}}\right|$ for any two points, $\mathcal{P}$ and $\mathcal{Q}$, at the same radius.
- Combining these results, we conclude that the light orbit is symmetric relative to the straight line $\overline{\mathrm{O}\mathcal{N}}$ in the SdS_w spacetime.

Calculating the usual deflection angle

- Generally speaking, the traditional approaches for calculating the *usual deflection angle* are based on integrating the LOE rather than what we have done for the Gaussian deflection angle.
- For comparison, we would like to calculate the usual deflection angle by using the new techniques presented in this work.
- In gravitational lensing, the observer and source are usually located in the outer region with $r \gg r_{\text{cri}}$, which can be approximately thought of as a laboratory area. Hereafter, we ideally assume that the SdS_w metric reduces to be conformally flat in the outer region so that the contribution from this region to the deflection of light can be ignored.
- With a conformal transformation, the reduced metric can be written as

$$\mathrm{dS}^2 = \Omega^2(\bar{\mathrm{r}})(-\mathrm{d}\bar{\tau}^2 + \mathrm{d}\bar{\mathrm{r}}^2 + \bar{\mathrm{r}}^2\,\mathrm{d}\phi^2)$$

- In addition, the light ray needs to pass through the matter-dominated region, that is, $b \lesssim r_{\text{cri}}$. Otherwise, if $b \gg r_{\text{cri}}$, the light ray will travel along a physically straight line, just as the one traveling in the Minkowski spacetime; we can not detect the bending of light, and are therefore unable to exact the information about the influence of dark energy on the bending of light.

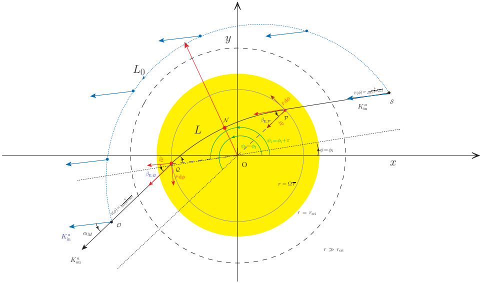


- As illustrated in the figure, we locate the source at point $(\bar{r}_{\mathcal{S}}, \phi_{\mathcal{S}})$ and the observer at point $(\bar{r}_{\mathcal{O}}, \phi_{\mathcal{O}})$, with $r_{\text{cri}} \ll \bar{r}_{\mathcal{O}}, \bar{r}_{\mathcal{S}} \ll r_o$. Here we use $\phi_{\mathcal{S}}$ and $\phi_{\mathcal{O}}$ to denote the polar angles of the source and observer, respectively.
- Since the source and observer are both far away from the mass center M sitting at origin O, the path of the incident ray from the source and that of the outgoing ray arriving at the position of the observer can be described well by two straight lines in the outer region, respectively.
- By the symmetry of the light orbit with respect to to the straight line $\overline{\mathrm{O}\mathcal{N}}$, the incident and outgoing rays should have the same impact parameter b. Thus, the two lines can be described by

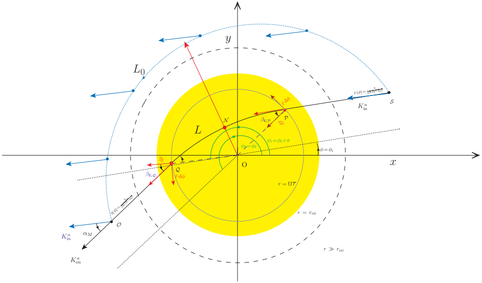
$$\bar{r} = \frac{b}{\cos(\phi - \phi_{\text{i}})} \quad \text{and} \quad \bar{r} = \frac{b}{\cos(\phi - \phi_{\text{f}})},$$

with ϕ_{i} and ϕ_{f} being the their polar angles from the positive x-axis, respectively. We define the two angles ϕ_{i} and ϕ_{f} concretely, as depicted in the above figure. In particular, we have $\phi_{\text{i}} \simeq \phi_{\mathcal{S}}$ and $\phi_{\text{f}} \simeq \phi_{\mathcal{O}}$ in the outer region $r \gg r_{\text{cri}} \gtrsim b$.

The incident and outgoing 4-momenta



- Denote $K_{\rm in}^{a}$ as the four-momentum vector of the incident light ray and $K_{\rm ou}^{a}$ as that of the outgoing one.
- They are both radial vectors so that their angular components $K_{\rm in}^{\phi}$ and $K_{\rm ou}^{\phi}$ are both zero: $K_{{\rm in}\perp}^{\;a} = (\Omega K_{\rm in}^{\bar{r}})\, e_{\bar{r}}^{a}$ with $\phi = \psi_{\rm i}$, and $K_{{\rm ou}\perp}^{\;a} = (\Omega K_{\rm ou}^{\bar{r}})\, e_{\bar{r}}^{a}$ with $\phi = \psi_{\rm f}$.
- Here, $\psi_{\rm i}$ and $\psi_{\rm f}$ are correspondingly the polar angles of these two vectors, respectively. In physics, they are just the incident and outgoing angles, respectively.
- Combining the definitions of $\phi_{\rm i}$ and $\phi_{\rm f}$, we have $\psi_{\rm i} = \phi_{\rm i} + \pi$ and $\psi_{\rm f} = \phi_{\rm f}$, as shown in the figure. In fact, both $\psi_{\rm i}$ and $\psi_{\rm f}$ are physically measurable angles; exactly, they are both measured from the x-axis that is actually the reference null geodesic with $b = 0$.



- The light ray travels along the path $L \doteq \mathcal{S} \to \mathcal{P} \to \mathcal{N} \to \mathcal{Q} \to \mathcal{O}$ passing through the matter-dominated region with $r \lesssim r_{\text{cri}}$.
- Recalling the parametrization for the γ curve, we take the parallel transport of $V^a(0) = K^a_{\text{in}}$ along the path L, and then get the corresponding vector $\bar{V}^a(\bar{\lambda}_0) = K^a_{\text{ou}}$ with $\bar{\varphi}(\bar{\lambda}_0) = \psi_{\text{f}}$ at point $\mathcal{O}$.
- We perform the parallel transport of $V^a(0) = K^a_{\text{in}}$ along L_0 in the outer area where $r \simeq \bar{\text{r}} \gg r_{\text{cri}}$, and obtain $V^a(\lambda_0) = K^a_{\text{in}}$ with $\varphi(\lambda_0) = \varphi(0)$ at point $\mathcal{O}$. Clearly, we have $\varphi(0) = \psi_{\text{i}} (= \phi_{\text{i}} + \pi)$.
- We therefore obtain the *usual deflection angle* α_M between the incident and outgoing rays: $\alpha_M = \bar{\varphi}(\bar{\lambda}_0) - \varphi(0) = \phi_{\text{f}} - \phi_{\text{i}} - \pi$, which is also physically measurable.
- With $u = 1/\text{r}$, integrating the LOE yields the following formula,

$$\alpha_M \simeq \phi_{\mathcal{O}} - \phi_{\mathcal{S}} - \pi = \sum_{p=\mathcal{S},\,\mathcal{O}} \int_{u_p}^{u_*} \frac{\mathrm{d}u}{\sqrt{1/b^2 - u^2 + 2Mu^3 + 2r_{\text{o}}^{3w+1} u^{3(w+1)}}} - \pi,$$

where $u_{p=\mathcal{S}} = 1/r_{\mathcal{S}}$, $u_{p=\mathcal{O}} = 1/r_{\mathcal{O}}$, and $u_* = 1/r_*$. Here, $r_{\mathcal{S}}$ and $r_{\mathcal{O}}$ are the radii of the source and observer in the original coordinates (r, ϕ), respectively.

- For the special case of the cosmological constant $(w = -1)$, by taking $u_{\mathcal{O}} \to 0^+$ and $u_{\mathcal{S}} \to 0^+$, we have $\alpha_M \simeq \frac{4M}{b}[1 + (\frac{b}{r_{\text{o}}})^2]$.