中国科学技术大学
University of Science and Technology of China

Gravitational EFTs

周双勇 (中科大 / 彭桓武高能基础理论研究中心)

第十二届新物理研讨会, 青岛, 2023年7月25日

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Gravitational EFTs and positivity bounds

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Outline

- Positivity bounds on EFT coefficients
- Einstein EFT
- Scalar-tensor EFTs

Positivity bounds

Snowmass White Paper: UV Constraints on IR Physics, de Rham, Kundu, Reece, Tolley & **SYZ**, 2203.06805

high energy UV theory
maybe unknown, but assume causality, unitarity, ...

Positivity bounds
(Causality bounds)

S-matrix bootstrap

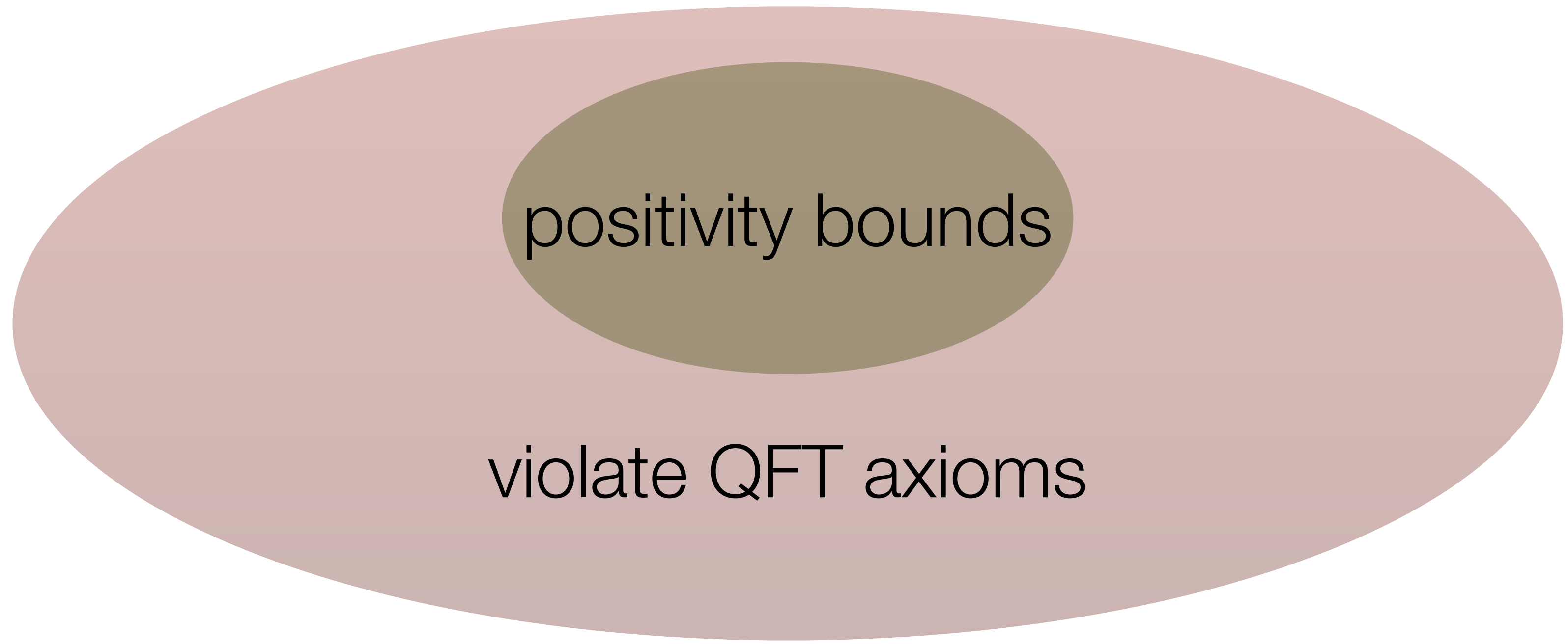
- Apply to generic LI EFTs
- For gravity, more subtle

low energy EFT
constraints on Wilson coefficients

these are not unitarity bounds!

Carving out EFT space

Naive EFT space
(Wilson coefficient space)



Simple example

Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi, 2006

$$\mathscr{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\lambda}{\Lambda^4}(\partial_\mu\phi\partial^\mu\phi)^2 + \cdots$$

$$A(s, t=0) = \cdots + \frac{2\lambda s^2}{\Lambda^4} + \cdots$$

"First" positivity bound: $\lambda > 0$

$$\mathscr{L}_{\mathrm{DBI}} \sim -\sqrt{1 + (\partial\phi)^2}$$ ✓

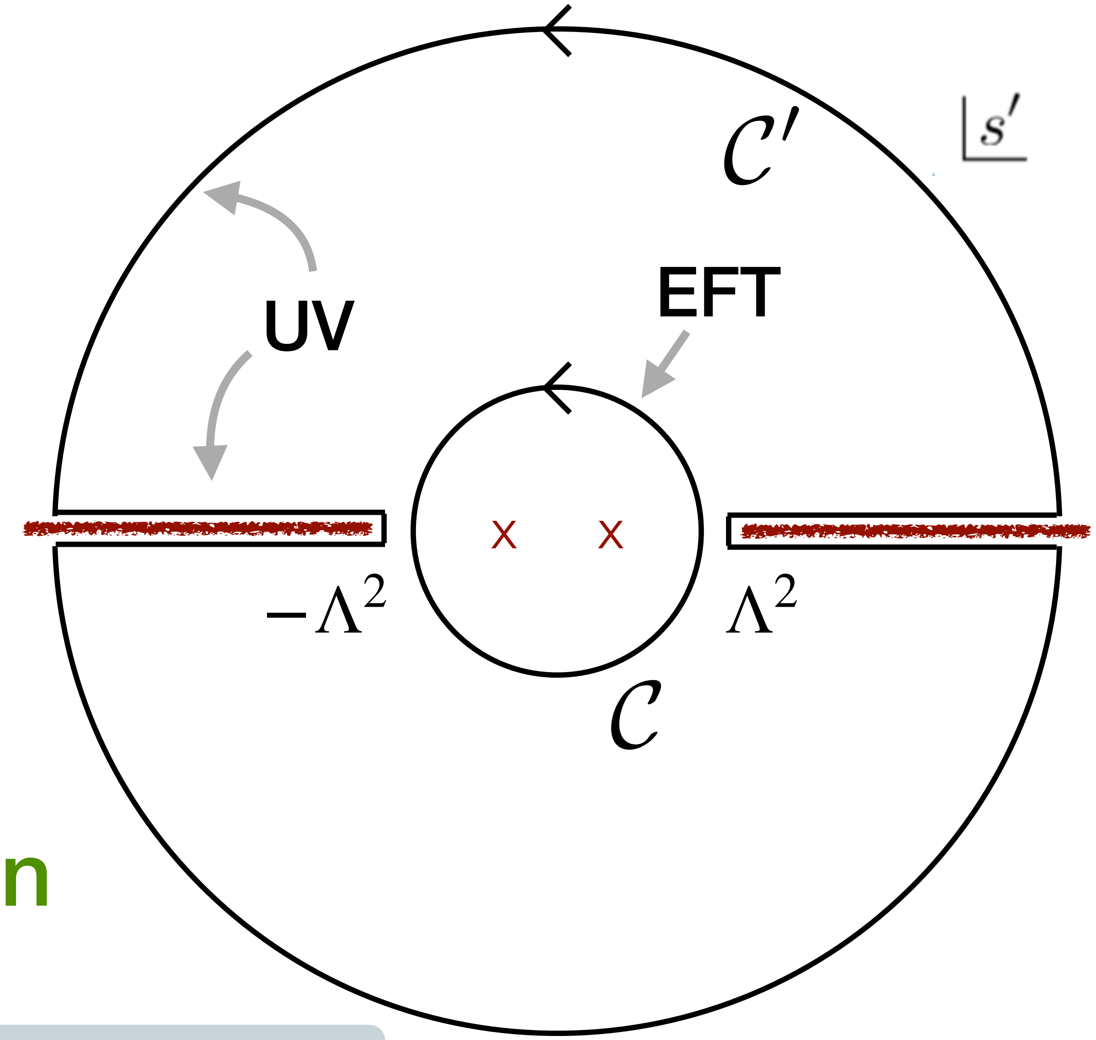
$$\mathscr{L}_{\overline{\mathrm{DBI}}} \sim \sqrt{1 - (\partial\phi)^2}$$ ✗

Dispersion relation (causality implies analyticity)

- Analyticity in complex s plane (fixed t)

$$A(s,t) = \frac{1}{2\pi i}\oint_{\mathcal{C}} ds' \, \frac{A(s',t)}{s'-s}$$

- Froissart bound $|A(s' \to \infty, t)| < s'^{2-\epsilon}$
- su crossing symmetry $A(s,t) = A(u,t)$



Twice subtracted dispersion relation

$$A(s,t) \sim \int_{\Lambda^2}^{\infty} \frac{\mathrm{d}\mu}{\pi\mu^2}\left[\frac{s^2}{\mu - s} + \frac{u^2}{\mu - u}\right] \mathrm{Im}\, A(\mu, t)$$

UV unitarity

$\mathrm{Im}\, a_\ell(\mu) > 0$

EFT amplitude **IR/UV connection** **UV full amplitude**

Two-sided bounds using full crossing symmetry

$$A(s,t) \sim c_{2,0}s^2 + c_{2,1}s^2t + c_{2,2}s^2t^2 + \cdots$$

Tolley, Wang & **SYZ**, 2011.02400
Caron-Huot & Duong, 2011.02957

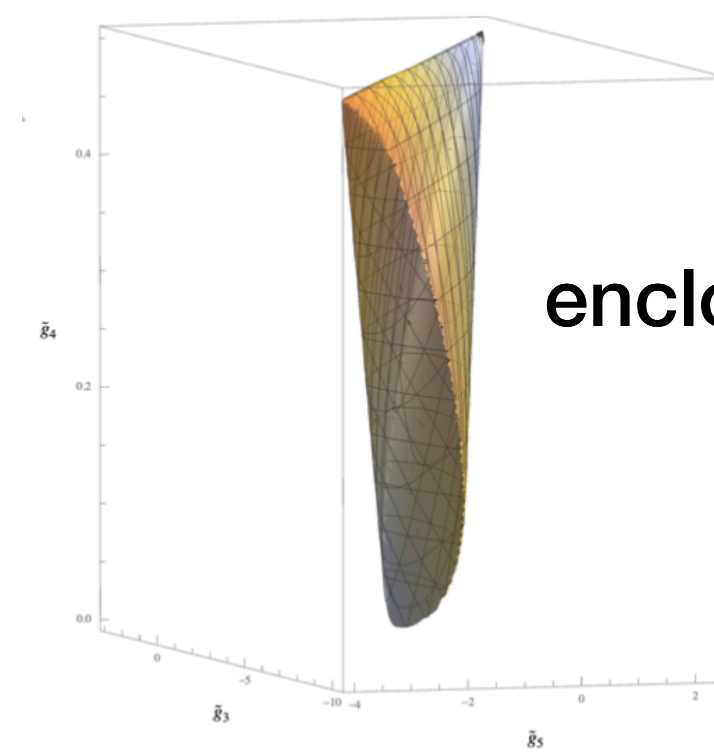
All Wilson coefficients are parametrically $\lesssim O(1)$!

Example of single scalar theory

(m,n)	Lower bounds	Upper bounds
$(1,1)$	$c_{1,1} > -\frac{3}{2}\sqrt{c_{1,0}c_{2,0}}$	$c_{1,1} < 8\sqrt{c_{1,0}c_{2,0}}$
$(2,1)$	$c_{2,1} > -\frac{5}{2}\sqrt{c_{2,0}c_{3,0}}$	$c_{2,1} < \frac{465}{38}\sqrt{c_{2,0}c_{3,0}}$
$(2,2)$	$c_{2,2} > -\frac{9}{2}c_{3,0}$	$c_{2,2} < \frac{2961}{58}c_{3,0}$
$(3,1)$	$c_{3,1} > -\frac{7}{2}\sqrt{c_{3,0}c_{4,0}}$	$c_{3,1} < \frac{1097}{58}\sqrt{c_{3,0}c_{4,0}}$
$(3,2)$	$c_{3,2} > -7c_{4,0}$	$c_{3,2} < \frac{10027}{59}c_{4,0}$
$(3,3)$	$c_{3,3} + \frac{3}{4}c_{4,1} > -\frac{147}{8}\sqrt{c_{4,0}c_{5,0}}$, $c_{3,3} - 8c_{4,1} > -154\sqrt{c_{4,0}c_{5,0}}$, $c_{3,3} -$ [illegible] $c_{4,1} > -$ [illegible] $\sqrt{c_{4,0}c_{5,0}}$, $c_{3,3} - 104c_{4,1} > -3369\sqrt{c_{4,0}c_{5,0}}$	$c_{3,3} - \frac{650}{41}c_{4,1} < -\frac{2310}{41}\sqrt{c_{4,0}c_{5,0}}$
$(4,2)$	$c_{4,2} > -\frac{17}{2}c_{5,0}$	$c_{4,2} < \frac{3923}{12}c_{5,0}$
$(4,3)$	$c_{4,3} + \frac{3}{4}c_{5,1} > -\frac{253}{8}\sqrt{c_{5,0}c_{6,0}}$, $c_{4,3} - \frac{180}{41}c_{5,1} > -\frac{8705}{82}\sqrt{c_{5,0}c_{6,0}}$, $c_{4,3} - \frac{325}{12}c_{5,1} > -\frac{16825}{24}\sqrt{c_{5,0}c_{6,0}}$, $c_{4,3} - \frac{169}{2}c_{5,1} > -\frac{11187}{4}\sqrt{c_{5,0}c_{6,0}}$ $c_{4,3} - \frac{743}{4}c_{5,1} > -\frac{63279}{8}\sqrt{c_{5,0}c_{6,0}}$	$c_{4,3} - \frac{73153}{1748}c_{5,1} < -\frac{708543}{3496}\sqrt{c_{5,0}c_{6,0}}$
$(4,4)$	$c_{4,4} + \frac{25}{24}c_{5,2} > -\frac{147}{8}c_{6,0}$, $c_{4,4} - \frac{125}{37}c_{5,2} > -\frac{71175}{74}c_{6,0}$, $c_{4,4} - \frac{785}{52}c_{5,2} > -\frac{83490}{13}c_{6,0}$, $c_{4,4} - \frac{2485}{69}c_{5,2} > -\frac{1144125}{46}c_{6,0}$	$c_{4,4} - 15c_{5,2} < -\frac{195}{2}c_{6,0}$, $c_{4,4} + \frac{368085}{36544}c_{5,2} < -\frac{2365845}{18272}c_{6,0}$

both upper and lower bound

assuming EFT is weakly coupled



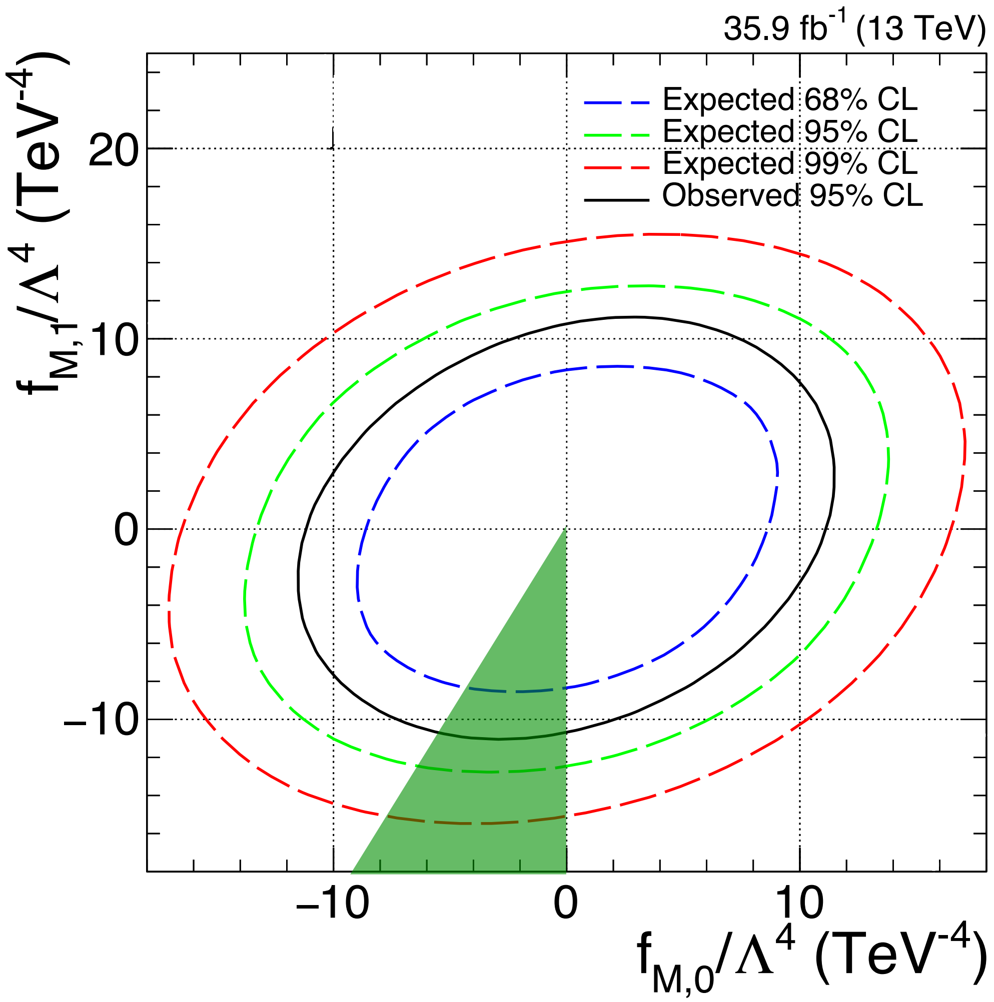
enclosed region

naturalness/dimensional analysis is a rigorous QFT theorem

Positivity bounds for SMEFT

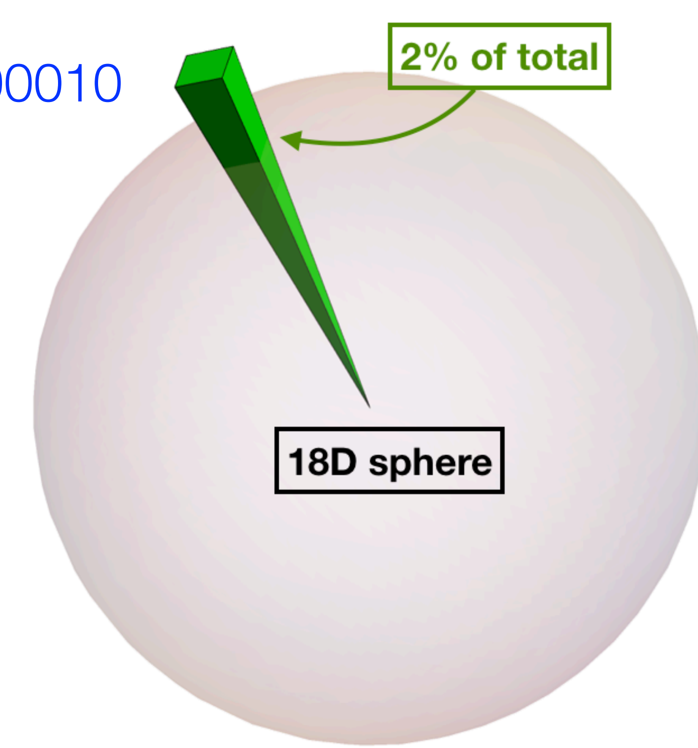
Consider VBS $V_1 + V_2 \to V_3 + V_4, \quad V_i \in \{Z, W^+, W^-, \gamma\}$

O_{M0} and O_{M1}



Space of 18 Wilson coeff's for aQGCs

Zhang & **SYZ**,1808.00010

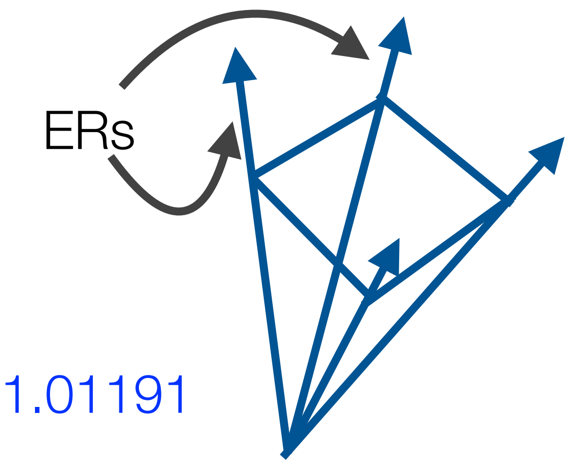


Only **<2%** of the total aQGC parameter space admits an analytic UV completion!

See also: Rodd & Remmen, 2004.02885; Gu, Wang & Zhang, 2011.03055; Li & Zhou, 2202.12907; …

Positivity cone and the inverse problem

Best s^2 bounds form a positivity cone



Extremal Ray ⟷ **UV Particle**

Zhang & **SYZ**, 2005.03047; Li, Xu, Yang, Zhang & **SYZ**, 2101.01191

Example: Higgs positivity cone in SMEFT

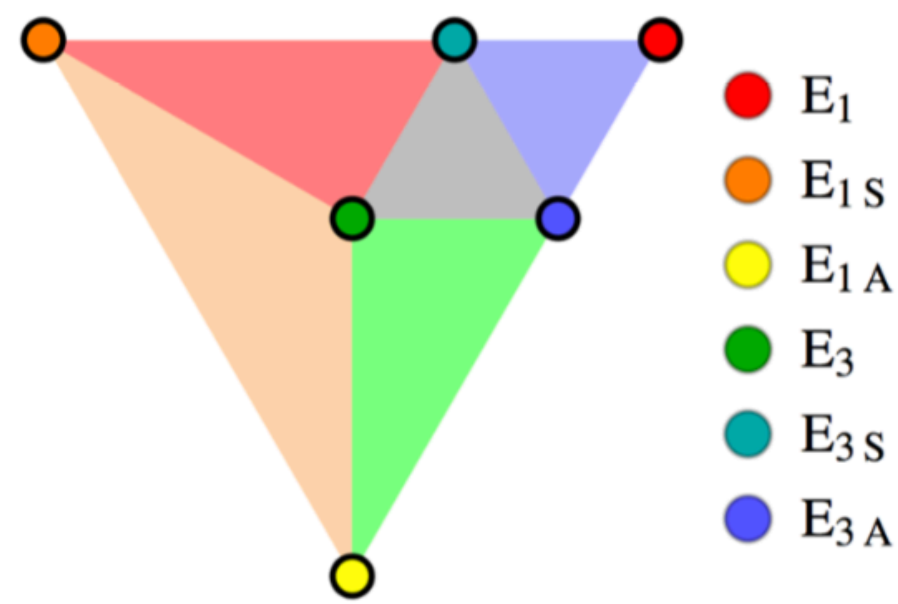
Wilson coeffs fall in blue region

↓

E_1 must exit

↓

new UV state ($SU(2)_L$ singlet, $Y = 1$)

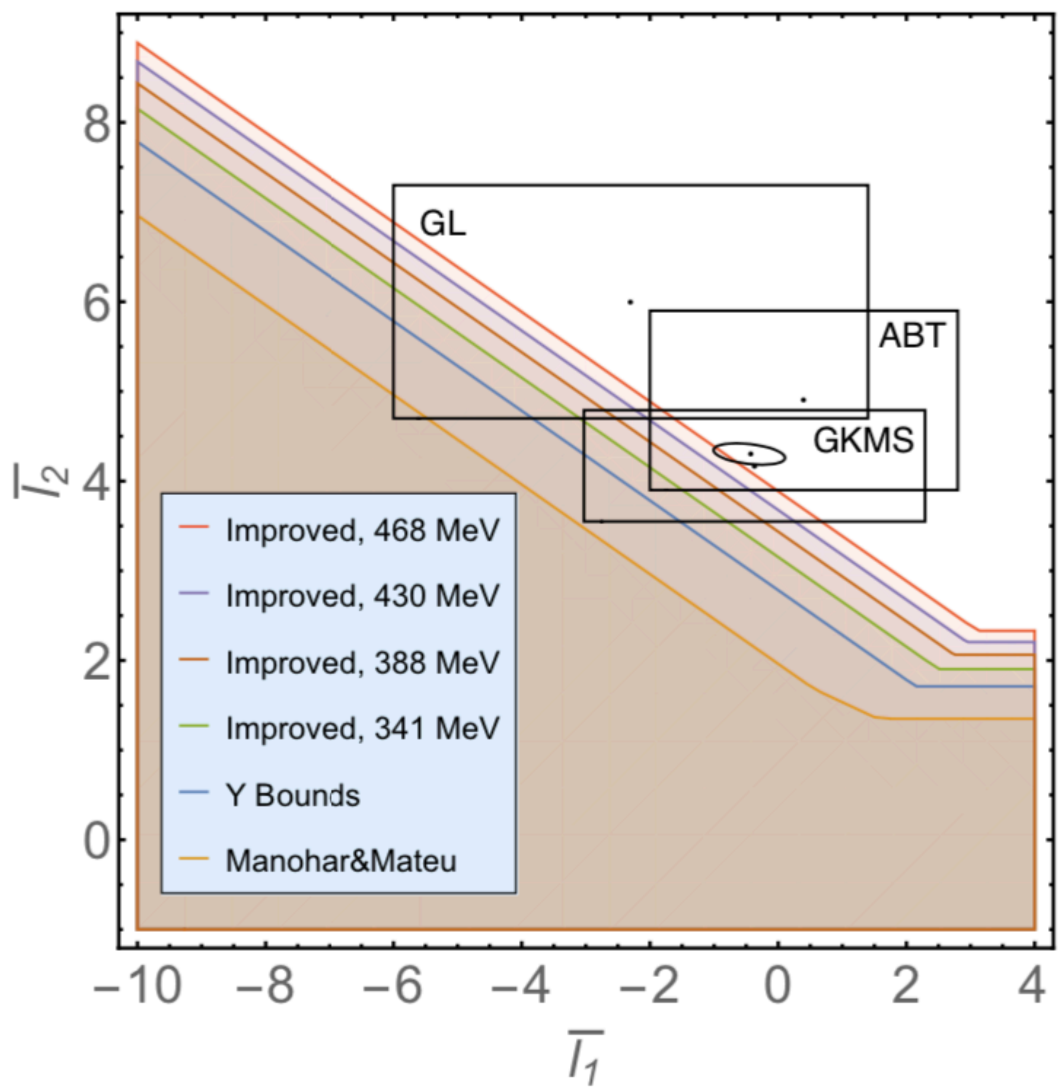


Positivity bounds (or dim-8 operators) are important to reverse-engineer UV physics!

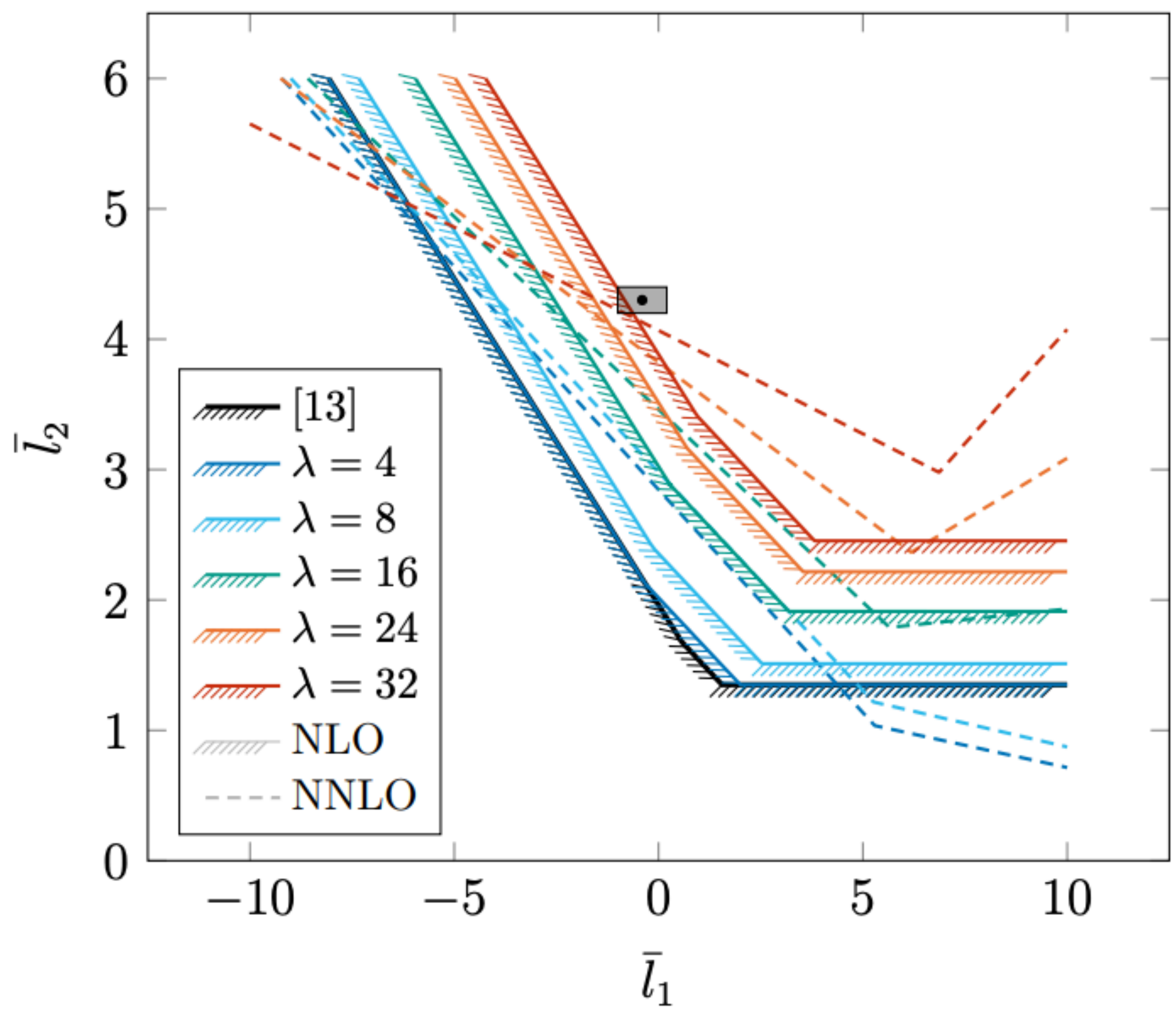
Zhang, 2112.11665

Positivity bounds for Chiral PT

For example, bounds on $O(p^4)$ coefficients



Wang, Feng, Zhang & **SYZ,** 2004.03992



Alvarez, Bijnens, Sjo, 2112.04253

See also: Manohar & Mateu, 0801.3222; Du, Guo, Meibner & Yao, 1610.02963
Guerrieri, Penedones & Vieira, 2011.02802

Einstein EFT

General relativity is an EFT

Around Minkowski space $\eta_{\mu\nu}$: $\quad g_{\mu\nu} = \eta_{\mu\nu} + \frac{2}{M_P} h_{\mu\nu}$

$h_{\mu\nu}$: massless spin-2 field $\Rightarrow$ graviton

weak-field limit: $|h_{\mu\nu}| \ll 1$

Einstein gravity $\quad \mathcal{L}_{EH} = \frac{M_P^2}{2}\sqrt{-g}R \sim \partial^2 h^2 + \frac{1}{M_P}\partial^2 h^3 + \frac{1}{M_P^2}\partial^2 h^4 + \cdots$

Gravity alone: EFT cutoff $\Lambda = M_P$

Symmetries: $\quad h'_{\rho\sigma} = \Lambda_\rho{}^\mu \Lambda_\sigma{}^\nu h_{\mu\nu} \qquad h'_{\mu\nu} = h_{\mu\nu} + 2\partial_{(\mu}\epsilon_{\nu)}$

cosmic acceleration universe	dark matter galaxy	solar system	strong gravity BH, NS, …	lab tests
10^{-33}eV	10^{-27}eV	10^{-18}eV	10^{-11}eV	10^{-3}eV
				

Higher order corrections to GR

$$\mathcal{L}_{\rm grav.EFT} \sim \mathcal{L}_{EH} + \sum_{i,j>2} \frac{1}{M_P^{i+j-4}} \partial^i h^j$$

diff. invariance → field redef.

combine into curvature tensors

use leading Einstein equation $R_{\mu\nu} = 0$

$$S = \frac{M_P^2}{2} \int \mathrm{d}^4x \sqrt{-g} \Big[R - \frac{1}{3!} \left(\alpha_3 R^{(3)} + \tilde{\alpha}_3 \tilde{R}^{(3)} \right) + \frac{1}{4} \left(\alpha_4 \left(R^{(2)} \right)^2 + \alpha_4' \left(\tilde{R}^{(2)} \right)^2 + 2\tilde{\alpha}_4 R^{(2)} \tilde{R}^{(2)} \right) + \ldots \Big]$$

$$R^{(2)} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}, \qquad \tilde{R}^{(2)} = R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma}, \qquad \tilde{R}_{\mu\nu\rho\sigma} \equiv \tfrac{1}{2} \epsilon_{\mu\nu}{}^{\alpha\beta} R_{\alpha\beta\rho\sigma}$$

$$R^{(3)} = R_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma}{}^{\alpha\beta} R_{\alpha\beta}{}^{\mu\nu}, \quad \tilde{R}^{(3)} = R_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma}{}^{\alpha\beta} \tilde{R}_{\alpha\beta}{}^{\mu\nu}$$

Integrating out DoFs

EFT cutoff $\Lambda \sim$ mass scale of lightest particle

A tale of 3 EFT theorists

Two scales in Einstein EFT: $M_P, \ \Lambda$

How do we do power counting?

Caron-Huot, Li, Parra-Martinez & Simmons-Duffin, 2201.06602

$$\sim \partial^2 h^2 + \frac{O(1)}{\Lambda^5} \partial^6 h^3$$

$$\mathcal{L} \sim M_P^2 R + \frac{O(1) M_P^3}{\Lambda^5} R^{(3)}$$

too relaxed

basically ignores M_P

$$\mathcal{L} \sim M_P^2 R + \frac{O(1)}{\Lambda^2} R^{(3)}$$

too restrictive

string theory violates it

$$\mathcal{L} \sim M_P^2 \left(R + \frac{O(1)}{\Lambda^4} R^{(3)} \right)$$

suggested positivity bounds!

correction < GR

Einstein EFT Wilson coefficients: $\alpha_3, \tilde{\alpha}_3 \sim \dfrac{1}{\Lambda^4}, \quad \alpha_4, \alpha_4', \tilde{\alpha}_4 \sim \dfrac{1}{\Lambda^6}$

Deviation from GR in strong gravity regime?

- Gravitational waves from aLIGO
- Black hole shadows from EHT

→ Observe strong gravity regime for first time

Endlich, Gorbenko, Huang & Senatore, 1704.01590

Generalize Goldberger-Rothstein EFT to include h.d. corrections

$$S_{\rm eff} = \int d^4x\sqrt{-g}2M_{\rm pl}^2\left(-R + \frac{\mathcal{C}^2}{\Lambda^6} + \frac{\tilde{\mathcal{C}}^2}{\tilde{\Lambda}^6} + \frac{\tilde{\mathcal{C}}\mathcal{C}}{\Lambda_-^6} + \dots\right)$$

$$\mathcal{C} \equiv R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$$
$$\tilde{\mathcal{C}} \equiv R_{\alpha\beta\gamma\delta}\tilde{R}^{\alpha\beta\gamma\delta}$$

Consider $\Lambda, \tilde{\Lambda}, \Lambda_- \sim \mathrm{km}^{-1}$

- Easily pass weak gravity tests
- Likely for table-top tests

Match to:

$$S_{\text{ext. obj.}} = \int dt\left\{\left[m_1 + m_2 + \frac{1}{2}\mu(t)\mathbf{v}_{\rm rel}^2 - V(r(t))\right] + \frac{1}{2}Q_{ij}(t)R^{i0j0} - \frac{1}{3}J_{ij}(t)\epsilon_{jkl}R^{kli0} + \dots\right\}$$

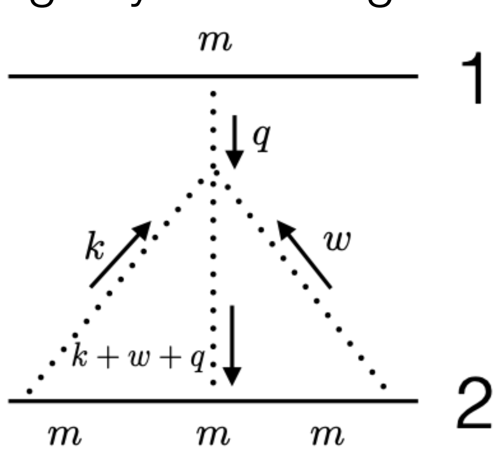
potential — quadrupole moment — current quadrupole moment

	Parity Even		Parity Odd
Operator	$\mathcal{C}^2$	$\tilde{\mathcal{C}}^2$	$\mathcal{C}\tilde{\mathcal{C}}$
$\Delta V/V$	v^4	$v^7 a$	$v^6 a$
$\Delta\omega/\omega$	v^4	$v^7 a$	$v^6 a$
$\Delta Q_{ij}/Q_{ij}$	v^4	$-$	v^5
$\Delta J_{ij}/J_{ij}$	$-$	v^4	v^3

	Parity Even		Parity Odd
Operator	$\mathcal{C}^2$	$\tilde{\mathcal{C}}^2$	$\mathcal{C}\tilde{\mathcal{C}}$
$\Delta P/P$	v^4	v^6	$v^6 a$
$\Delta h/h$	v^4	v^5	v^4

often at observable 2PN

eg Feynman diagram



Speed of gravity

de Rham and Tolley, 1909.00881

In *classical* GR — due to equivalence principle

In Minkowski vacuum $c_{\rm gw} = 1$ — On nontrivial background $c^{\rm (LB)}_{\rm gw} \leq 1$

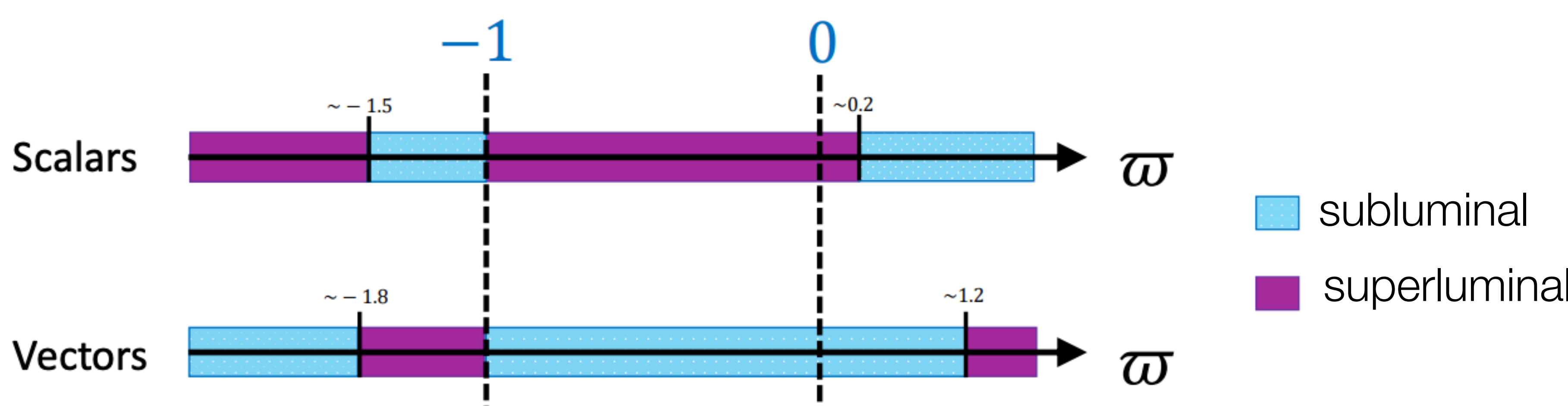
In Einstein EFT (with h.d. corrections)

On nontrivial background $c^{\rm (LB)}_{\rm gw} > 1$

allowed by positivity bounds
seen in loop UV example
or string theory example

even if background preserves null energy condition

eg, Integrate out matter loops & on FRW background



Similar to photon's velocity in curved space Drummond & Hathrell, 1980

Graviton t-channel pole

Spin-2 pole s^2/t survives twice subtraction

$$\frac{1}{M_P^2 t} + (\cdots) \sim \int_{\Lambda^2}^{\infty} \frac{\mathrm{d}\mu}{\mu^3} \mathrm{Im}\, A(\mu, t)(\cdots)$$

Bounds are not strictly positive

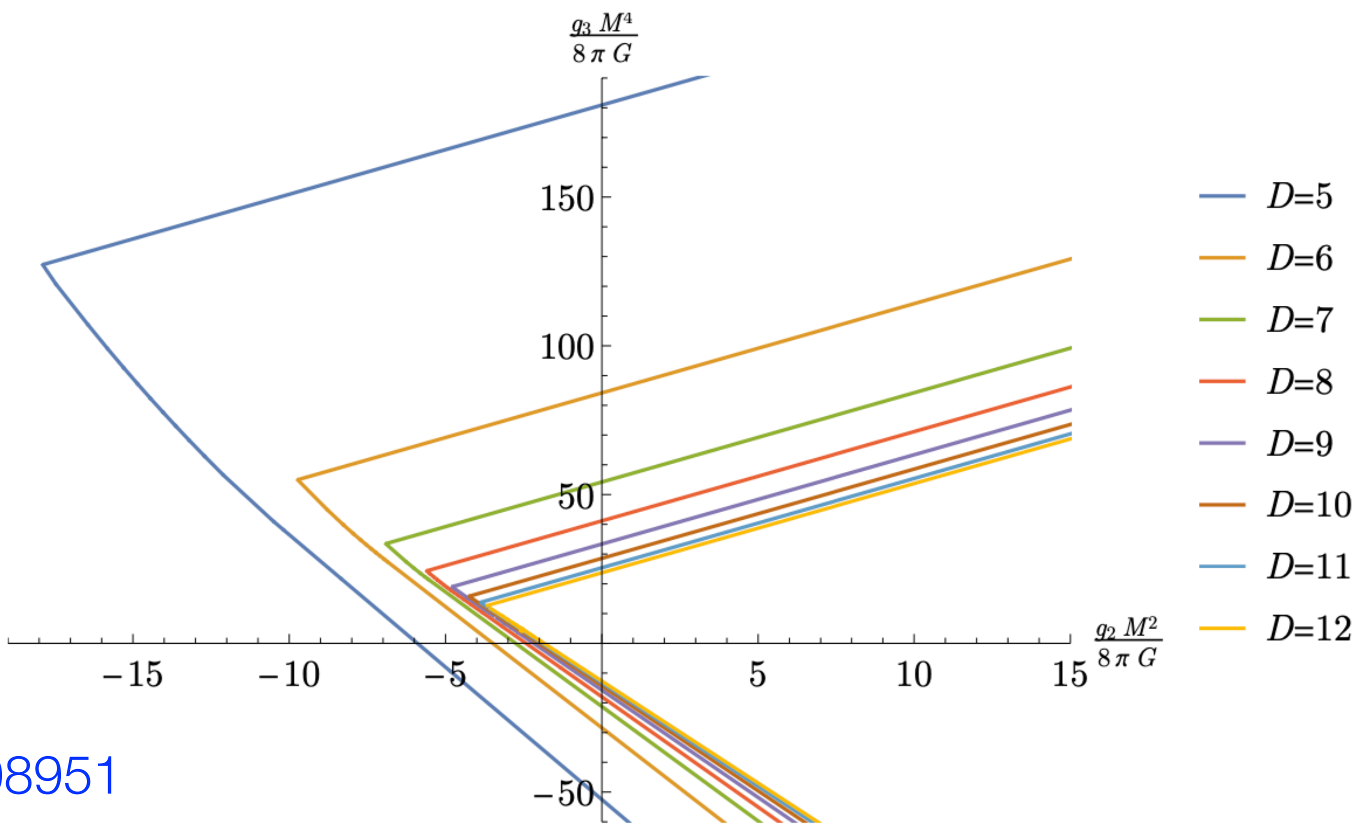
Alberte, de Rham, Jaitly & Tolley, 2007.12667
Tokuda, Aoki & Hirano, 2007.15009

$$a_{2,0} > -\frac{\Lambda^2}{M_{\mathrm{Pl}}^2} \times \mathcal{O}(1)$$

Numerical bounds

functional optimization

impact parameter $b = \ell/\mu$



Caron-Huot, Mazac, Rastelli, Simmons-Duffin,2102.08951

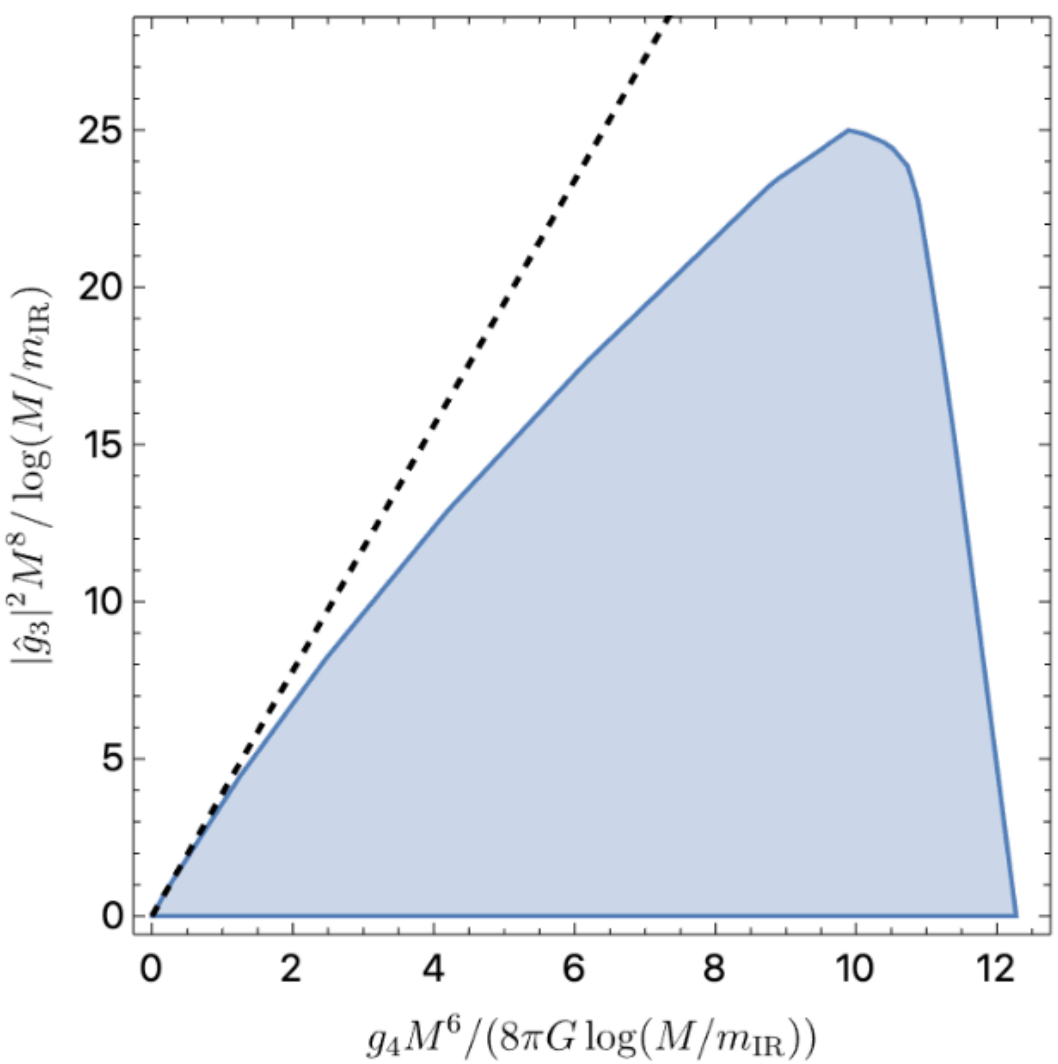
Positivity bounds on Einstein EFT

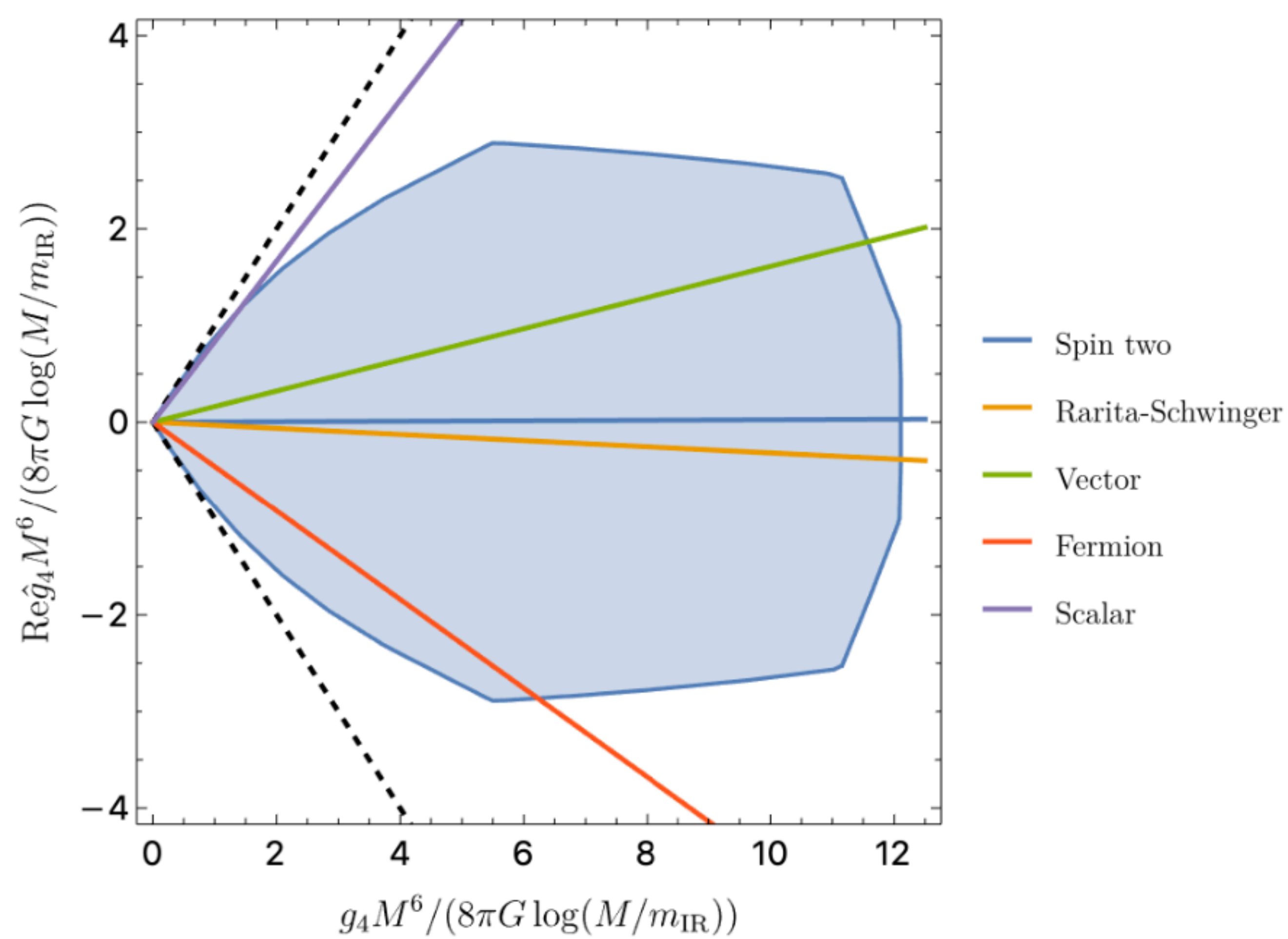
Massless graviton

Caron-hot, Li, Parra-Martinez & Simmons-Duffin, 2201.06602
Chiang, Huang, Li, Rodina & Weng, 2201.07177

IR divergence ⇒ introduce cutoff $m_{\rm IR}$

project bounds to $\log(\Lambda/m_{\rm IR}) \sim O(10)$





Low spin dominance

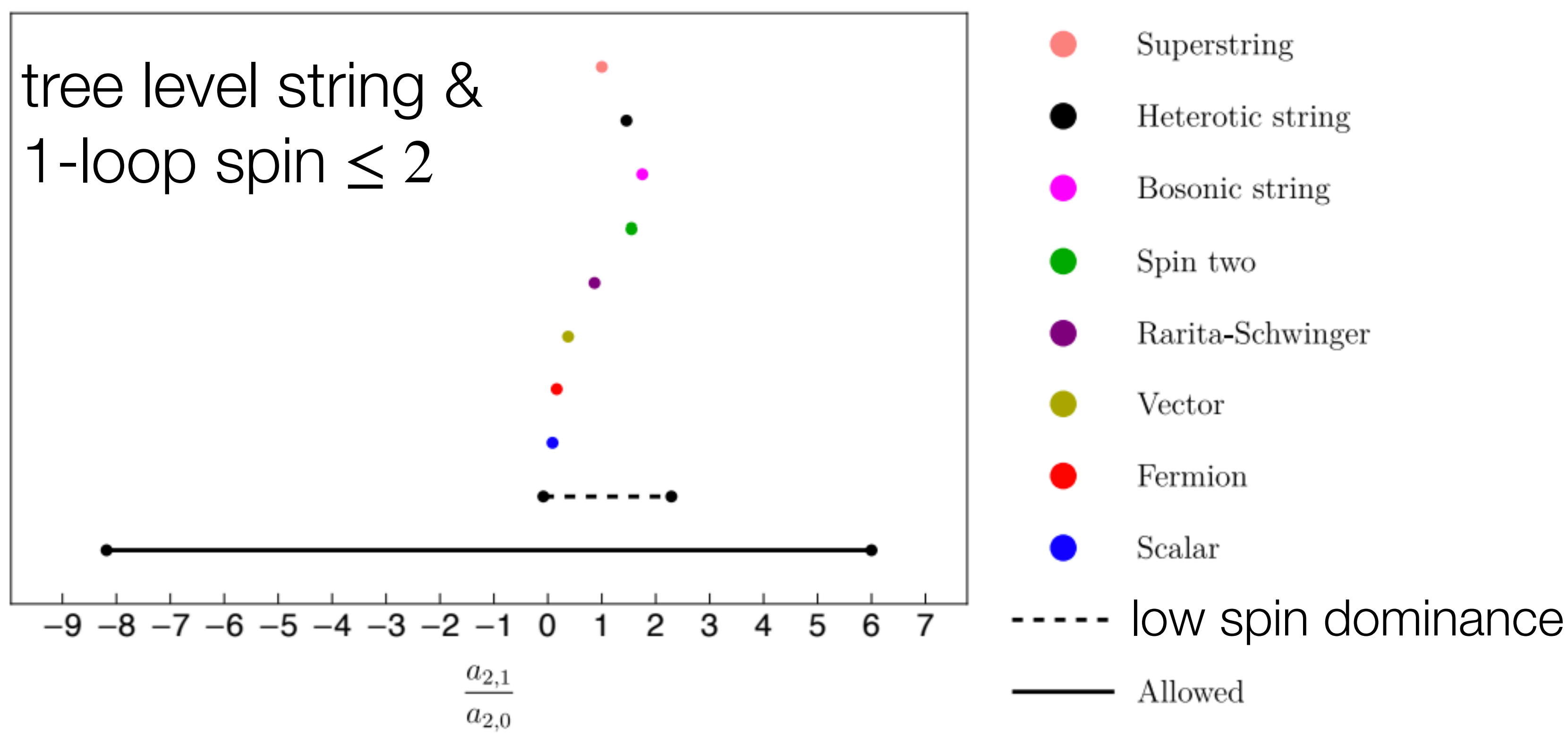
Bern, Kosmopoulosa & Zhiboedov, 2103.12728

Assume spectral density

$$\langle \rho_4^{+-} \rangle_k \geq \alpha \langle \rho_{J>4}^{+-} \rangle_k$$

$$\langle \rho_0^{++} \rangle_k \geq \alpha \langle \rho_{J>0}^{++} \rangle_k$$

choose by-hand $\alpha = 10^2$



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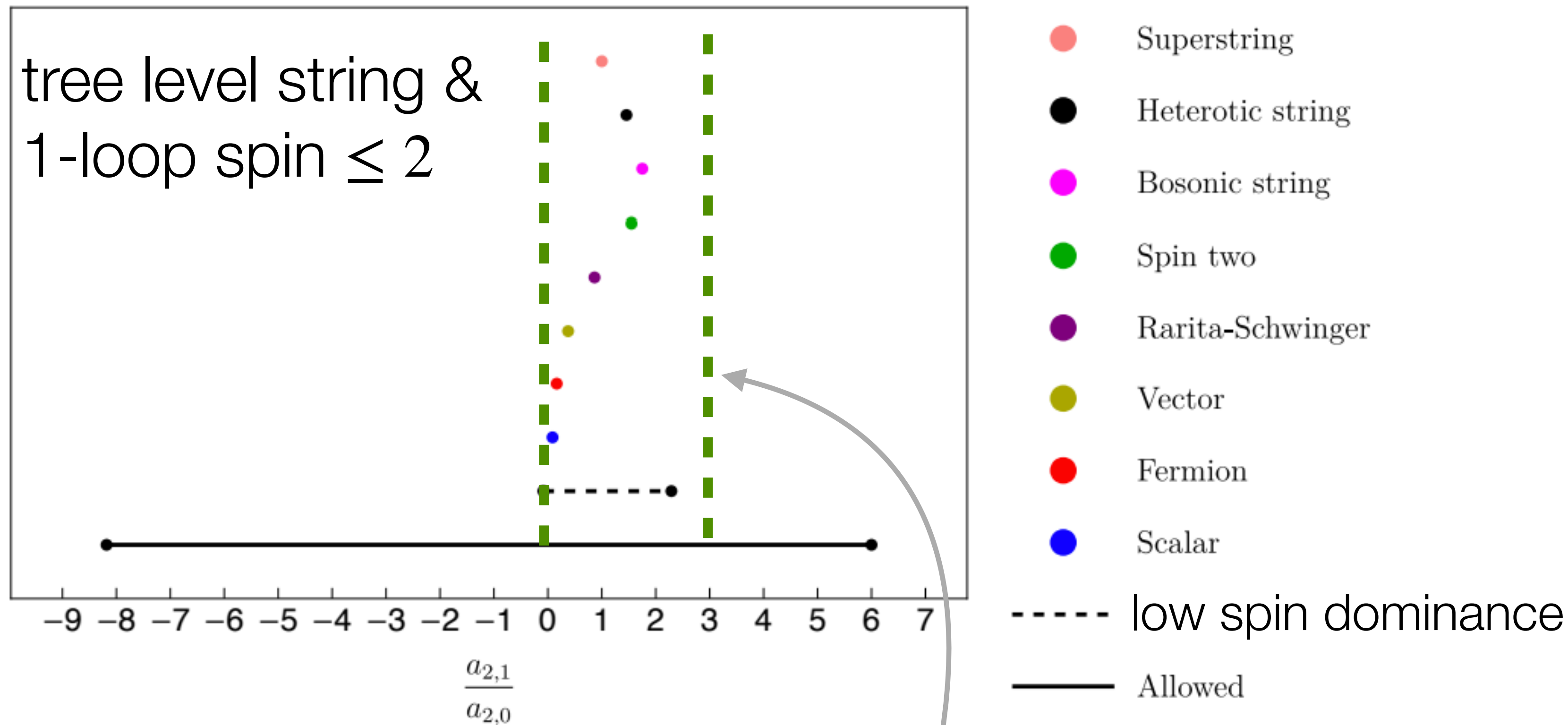
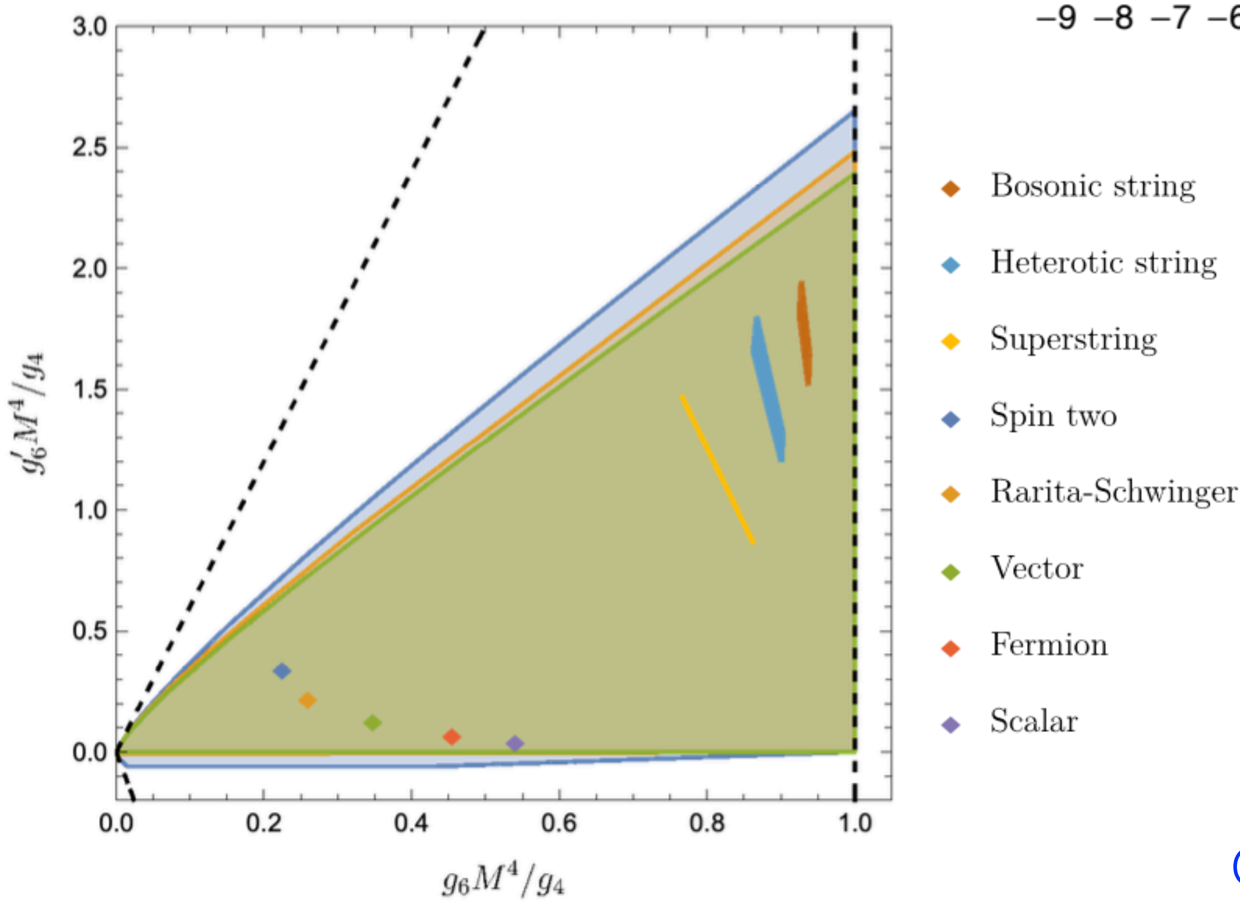




positivity bounds by Caron-Hot *et al*

no need to assume
low spin dominance!

Bern et al used weak positivity bounds

Caron-hot, Li, Parra-Martinez & Simmons-Duffin, 2201.06602

Einstein-Maxwell EFT

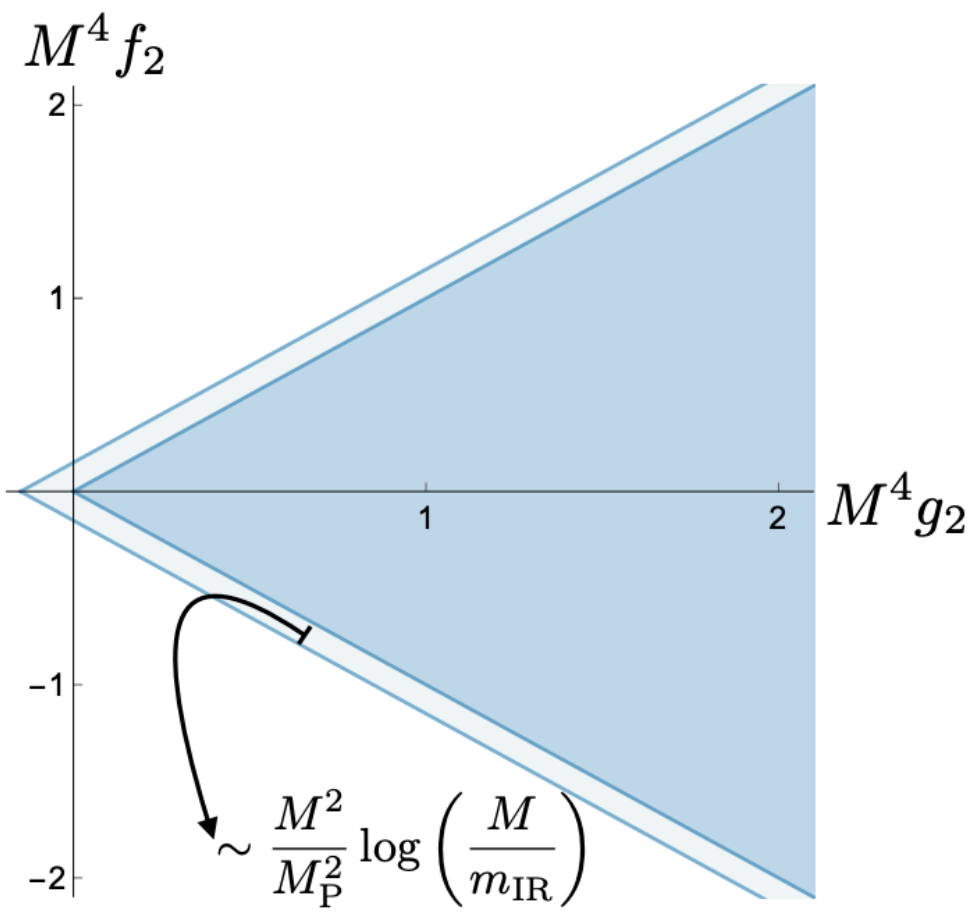
At lowest energies

known massless particles: graviton & photon

$$\mathcal{L} = \sqrt{-g}\left(\frac{M_\mathrm{P}^2}{2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \alpha_1(F_{\mu\nu}F^{\mu\nu})^2 + \alpha_2\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2 + \beta W_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} + \ldots\right)$$

Weak gravity conjecture $q \geqslant \dfrac{m}{\sqrt{2}M_\mathrm{P}}$ or $4\alpha_1 \pm \frac{1}{M_\mathrm{P}^2}\beta > 0, \ \ \alpha_2 > 0$

WGC implied by positivity bounds if t-channel pole is ignorable



Cheung & Remmen, 1407.7865; Noumi & Shiu, 1810.03637
Bellazzini, Lewandowski & Serra, 1902.03250

However, t-channel pole is not ignorable

Alberte, de Rham, Jaitly & Tolley, 2007.12667, 2012.05798

Explicitly compute with t-channel pole

Positivity bounds do not imply weak gravity conjecture

Henriksson, McPeak, Russo & Vichi, 2203.08164

Causality bounds within EFT

Shapiro time delay (around a BH)

EFT corrections

$$\frac{\mathrm{d}^2\Psi^{\pm}_{\omega\ell}}{\mathrm{d}r_*^2} = -\left[\omega^2 - V^{\pm}_{\mathrm{GR}}(r;\,\ell) - c_1\mu\, V^{\pm}(r;\,\ell,\,\omega)\right]\Psi^{\pm}_{\omega\ell}\,,$$

At infinity $r_* \to -\infty$ $\quad \Psi_\ell \propto e^{2i\delta_\ell} e^{i\omega r_*} - (-1)^\ell e^{-i\omega r_*}$

Time delay $\quad T_\ell = 2\frac{\partial\delta_\ell(\omega)}{\partial\omega} \equiv T_\ell^{\mathrm{GR}} + T_\ell^{\mathrm{EFT}}$

often $T_\ell^{\mathrm{EFT}} < 0$
ie, time advance

- Asymptotic causality $-T_\ell < \omega^{-1}$
- Infrared causality $-T_\ell^{\mathrm{EFT}} < \omega^{-1}$

eg, Camanho, Edelstein, Maldacena & Zhiboedov, 1407.5597

de Rham, Tolley & Zhang, 2112.05054

Example $\quad S^{(1)}_{\mathrm{D8}} = \int \mathrm{d}^4x\sqrt{-g}\,\frac{M_{\mathrm{Pl}}^2}{2}\left[R + \frac{c_1}{\Lambda^6}\left(R_{abcd}R^{abcd}\right)^2\right]$

Infrared causality for $3M_\odot$ BH $\Rightarrow\ \Lambda \gtrsim 7\times 10^{-11}\mathrm{eV}$

but current GW experiments can only reach $\Lambda \sim 10^{-13}\mathrm{eV}$

Quantum corrections

Feynman, 1963; De Witt, 1967; Donoghue, gr-qc/9405057

Wrong statements often seen in popular science:

- "We can't quantize general relativity!"
- "General relativity is incompatible with quantum mechanics."
- "General relativity is non-renormalizable, so we can't quantize it."
- …

We can quantize general relativity.

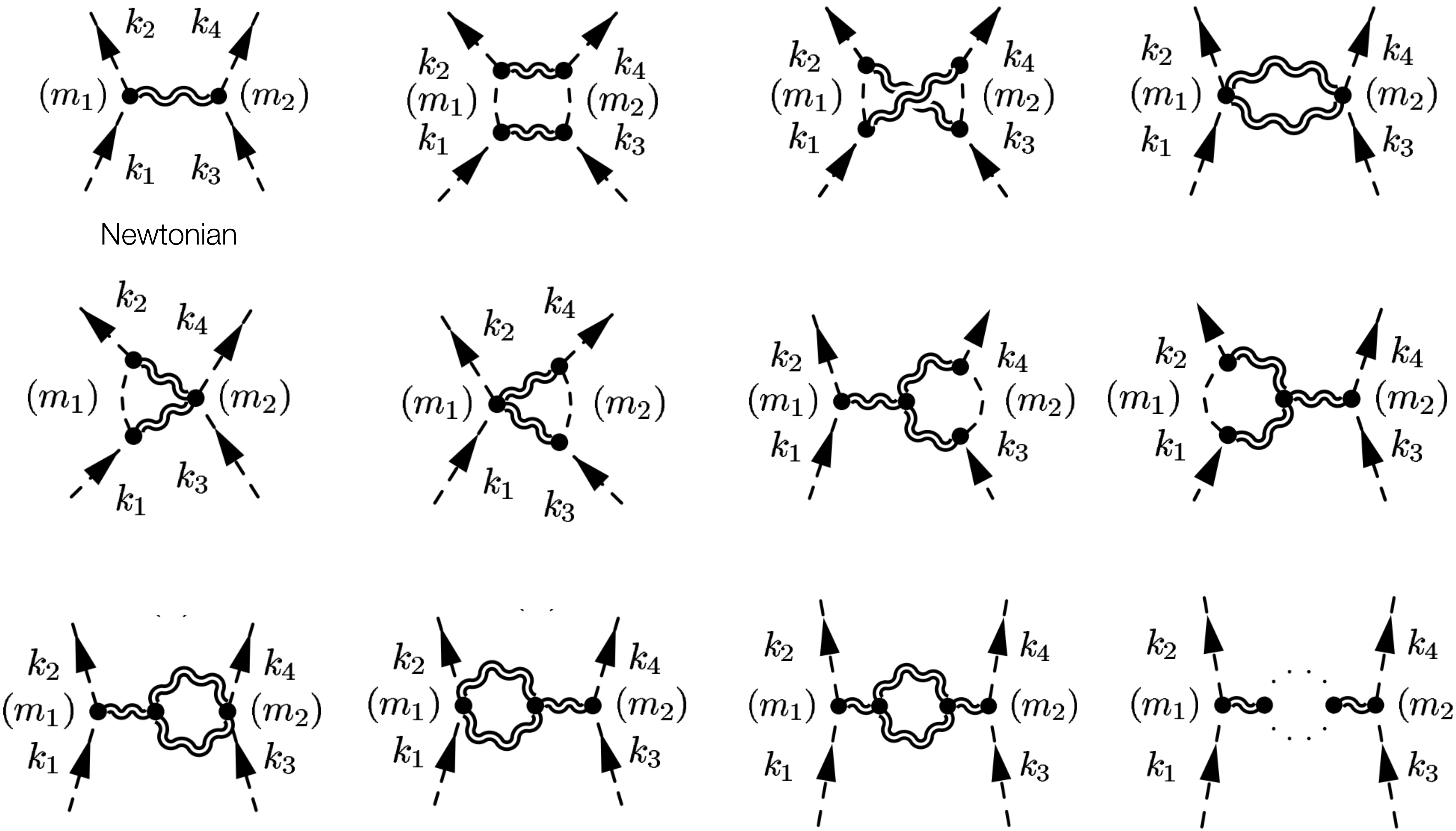
Example: quantum correction to Newton's law

$$V(r) = -\frac{Gm_1m_2}{r}\left[1 + 3\frac{G(m_1+m_2)}{r} + \frac{41}{10\pi}\frac{G\hbar}{r^2}\right]$$

Bjerrum-Bohr, Donoghue and Holstein, hep-th/0211072

quantum correction

Compute potential from Feynman diagrams



Bjerrum-Bohr, Donoghue and Holstein, hep-th/0211072

Scalar-tensor EFTs

Scalar-tensor theory

Theory with light DoFs: $g_{\mu\nu}$ + (real) scalar ϕ

not Higgs

Cosmologists like scalar fields!

- inflaton
- dark energy
- dilaton, axion
- …

- easy to be consistent with cosmo. principle
- form classical configurations, unlike fermions
- many of them from string/M theory

Earliest example: Brans-Dicke theory ⇒ variable G_N

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(\phi R - \frac{\omega}{\phi}\partial_a\phi\partial^a\phi\right)$$

current bound $\omega \gtrsim 10^5$
by solar system tests of GR

More generally, a scalar-tensor EFT being constraints by observations

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big(\frac{M_P^2}{2}R - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \frac{\lambda_3}{3!}\phi^3 - \frac{\lambda_4}{4!}\phi^4 + \frac{\alpha}{2}(\nabla_\mu\phi\nabla^\mu\phi)^2 + \frac{\beta_1}{2!}\phi\mathcal{G} + \frac{\beta_2}{4}\phi^2\mathcal{G}$$
$$+ \frac{\gamma_0}{3!}\mathcal{R}^{(3)} + \frac{\gamma_1}{3!}\phi\mathcal{R}^{(3)} + \frac{\gamma_2}{2}\nabla_\mu\phi\nabla^\mu\phi\mathcal{R}^{(2)} - \frac{4\gamma_3}{3}\nabla_\mu\phi\nabla_\rho\phi\nabla_\nu\nabla_\sigma\phi R^{\mu\nu\rho\sigma} + \dots\Big)$$

Hairy black holes

BH uniqueness theorems in GR   Kerr BH is unique solution

No-hair theorems   Ruffini & Wheeler, 1971

uniqueness of BHs in presence of matter fields

Case of scalar field: a few no-go theorems

Hawking, 1972; Bekenstein, 1995
Sotiriou & Faraoni, 1109.6324
Hui & Nicolis, 1202.1296

But there are hairy BHs   Sotiriou & **SYZ**, 1312.3622

$$S = \frac{M_p^2}{2} \int d^4x \sqrt{-g} \left(R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \alpha \phi \mathcal{G} \right)$$

$\mathcal{G}$: Gauss-Bonnet invariant

from EFT viewpoint, easy to have hairy BHs: $\phi\mathcal{G}$ is leading coupling

Used as a fiducial model to

test deviations from GR in strong gravity regime (GWs, …)

Spontaneous scalarization

- GR solution with $\phi = const$ in weak gravity
- Deviates from GR only in strong gravity regime

Neutron stars

Damour & Esposito-Farese, 1995

$$S = \frac{1}{16\pi G_*} \int d^4x \sqrt{-g} [R - 2\nabla_\mu \varphi \nabla^\mu \varphi] + S_m \left[\Psi_m ; \mathcal{A}^2(\varphi) g_{\mu\nu} \right]$$

non-minimal coupling

Black holes

Doneva & Yazadjiev, 1711.01187

Silva, Sakstein, Gualtieri, Sotiriou & Berti, 1711.02080

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[R - \frac{1}{2} \nabla_\mu \varphi \nabla^\mu \varphi + f(\varphi) \mathcal{G} \right]$$

$f(\varphi) = a\varphi^2 + b\varphi^4 + \ldots$ no linear term, so not always hairy

Both of them rely on tachyonic instability in scalar sector

$$\Box\, \delta\varphi + m^2(\mathcal{G}, T)\, \delta\varphi + \ldots = 0$$

$m^2 > 0$ stay in GR solution

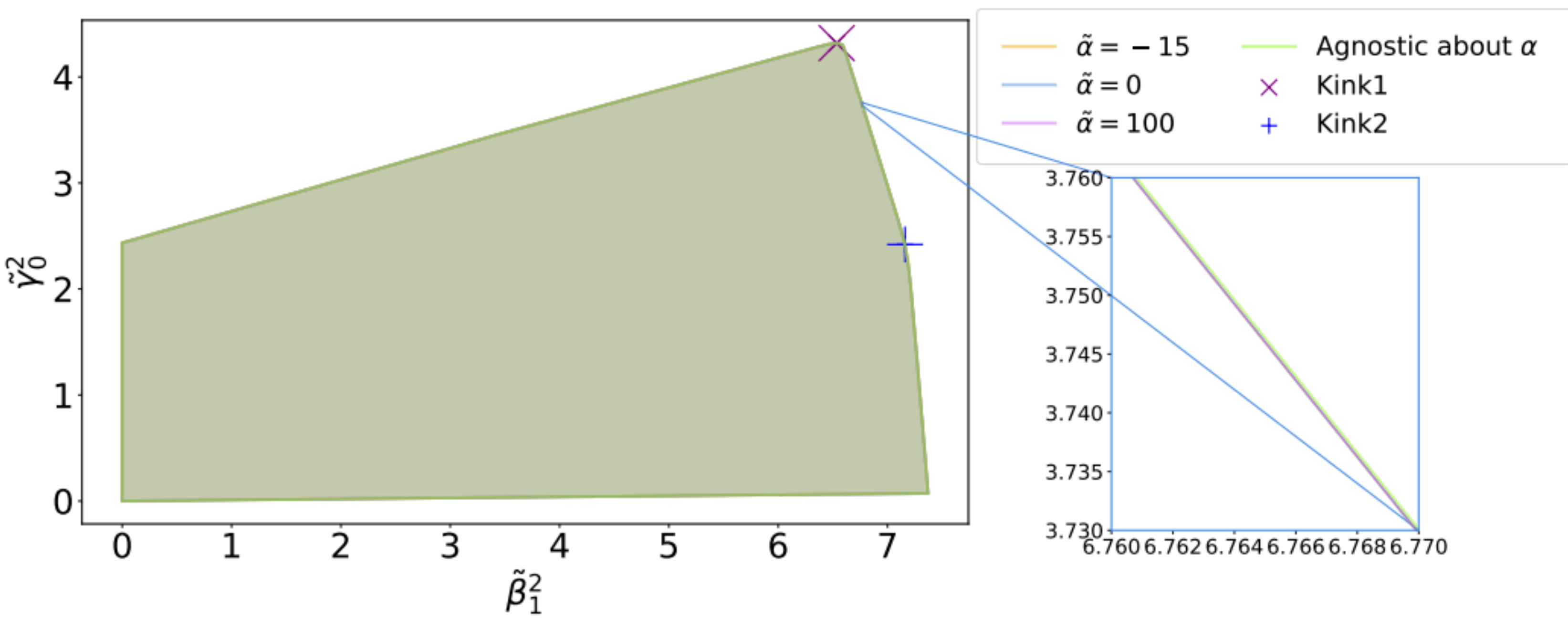
$m^2 < 0$ roll down to hairy solution

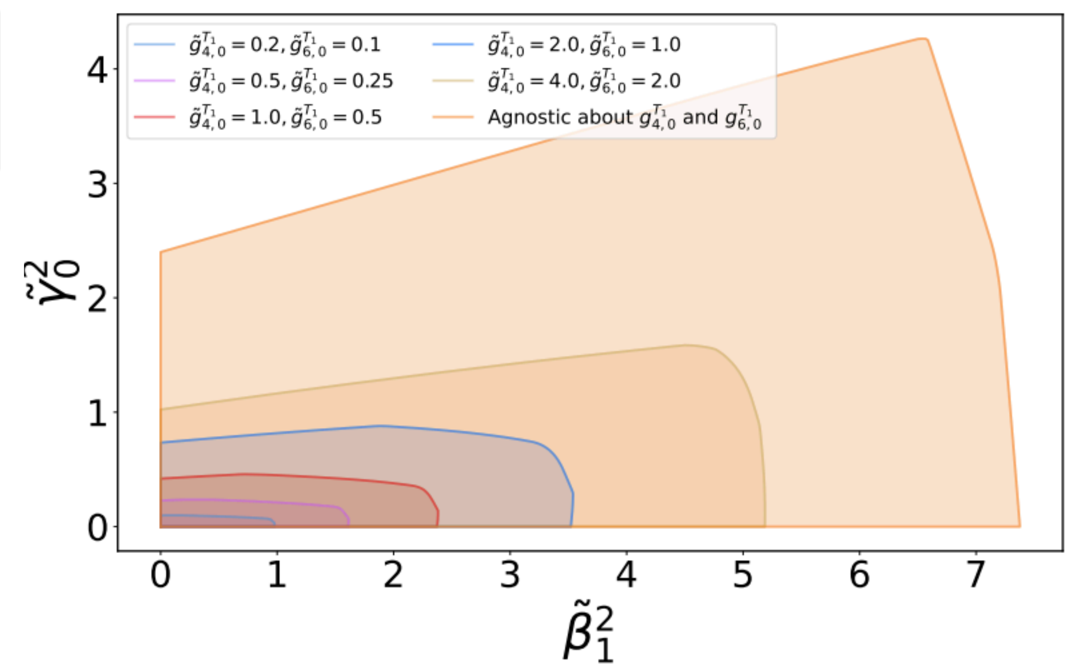
Positivity bounds on scalar-tensor EFT

Scalar-tensor EFT

Hong, Wang, **SYZ**, 2304.01259

$$S=\int \mathrm{d}^4x\sqrt{-g}\Big(\frac{M_P^2}{2}R-\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi-\frac{\lambda_3}{3!}\phi^3-\frac{\lambda_4}{4!}\phi^4+\frac{\alpha}{2}(\nabla_\mu\phi\nabla^\mu\phi)^2+\frac{\beta_1}{2!}\phi\mathcal{G}+\frac{\beta_2}{4}\phi^2\mathcal{G}$$
$$+\frac{\gamma_0}{3!}\mathcal{R}^{(3)}+\frac{\gamma_1}{3!}\phi\mathcal{R}^{(3)}+\frac{\gamma_2}{2}\nabla_\mu\phi\nabla^\mu\phi\mathcal{R}^{(2)}-\frac{4\gamma_3}{3}\nabla_\mu\phi\nabla_\rho\phi\nabla_\nu\nabla_\sigma\phi R^{\mu\nu\rho\sigma}+\dots\Big)$$





$\phi\mathcal{G}$ and $\phi^2\mathcal{G}$ generate hair BHs and spontaneous scalarization

$$\mathcal{L}\supset M_P^2\sqrt{-g}\Big(\frac{\mathcal{O}(1)}{\Lambda^2}\varphi\mathcal{G}+\frac{\mathcal{O}(1)M_P}{\Lambda^3}\varphi^2\mathcal{G}\Big)$$

scalarization is natural!

Power counting via dispersion relations

Use normalized dispersion relations

Hong, Wang, **SYZ**, 2304.01259

$$-\frac{1}{M_P^2 t} = \sum_{\ell,X} 16\pi(2\ell+1)\int_{\Lambda^2}^{\infty}\frac{\mathrm{d}\mu}{\pi}\left[\frac{d_{0,0}^{\ell,\mu,t}\left|c_{\ell,\mu}^{++}\right|^2}{\mu^3}+\frac{d_{4,4}^{\ell,\mu,t}\left|c_{\ell,\mu}^{+-}\right|^2}{(\mu+t)^3}\right] \Rightarrow \frac{\Lambda^2}{M_P^2} = -\hat{t}\sum_{\ell,X}\int_1^{\infty}\mathrm{d}\hat{\mu}\left[\frac{d_{0,0}^{\ell,\hat{\mu},\hat{t}}\left|\hat{c}_{\ell,\mu}^{++}\right|^2}{\hat{\mu}^3}+\frac{d_{4,4}^{\ell,\hat{\mu},\hat{t}}\left|\hat{c}_{\ell,\mu}^{+-}\right|^2}{(\hat{\mu}+\hat{t})^3}\right]$$

$c_{\ell,\mu}^{++}, c_{\ell,\mu}^{+-}$: UV partial amplitude $\quad d_{h_1,h_2}^{\ell,\mu,t}$: Wigner d-matrices

Find correspondences

$$\frac{\Lambda}{M_P} \Leftrightarrow \hat{c}_{\ell,\mu}^{++}, \hat{c}_{\ell,\mu}^{+-}, \hat{c}_{\ell,\mu}^{-+}, \hat{c}_{\ell,\mu}^{--} \qquad \frac{\Lambda}{M_P} \Leftrightarrow \hat{c}_{\ell,\mu}^{+0}, \hat{c}_{\ell,\mu}^{-0}, c_{\ell,\mu}^{0+}, \hat{c}_{\ell,\mu}^{0-} \qquad \begin{cases} 1 \Leftrightarrow \hat{c}_{\ell,\mu}^{00} \\ \frac{\Lambda}{M_P} \Leftrightarrow \hat{c}_{\ell,\mu}^{00} \end{cases}$$

eg, $$-\frac{\Lambda^6\gamma_1}{M_P^3} = \sum_{\ell,X}\int_1^{\infty}\mathrm{d}\hat{\mu}\left[\frac{(2\hat{\mu}-3\hat{t})d_{2,0}^{\ell,\hat{\mu},\hat{t}}\hat{c}_{\ell,\mu}^{+0}\hat{c}_{\ell,\mu}^{*,--}}{\hat{t}\,\hat{\mu}^4}-\frac{\hat{t}\,\partial_{\hat{t}}d_{0,-2}^{\ell,\hat{\mu},0}\hat{c}_{\ell,\mu}^{++}\hat{c}_{\ell,\mu}^{*,-0}}{\hat{\mu}^3(\hat{\mu}-\hat{t})}+\frac{\hat{t}\,\partial_{\hat{t}}d_{2,0}^{\ell,\hat{\mu},0}\hat{c}_{\ell,\mu}^{+0}\hat{c}_{\ell,\mu}^{*,--}}{\hat{\mu}^3(\hat{\mu}+\hat{t})}\right] \Rightarrow \gamma_1 \sim \frac{M_P}{\Lambda^4}$$

For lowest few orders

$$\widehat{\mathcal{O}}_{\phi R} \sim M_P^2\Lambda^2\left[\frac{\nabla}{\Lambda}\right]^{N_\nabla}\left[\frac{R}{\Lambda^2}\right]^{N_R}\left[\frac{\phi}{M_P}\right]^{N_\phi}\left[\frac{M_P}{\Lambda}\right]^{\tilde{N}_\phi} \qquad \tilde{N}_\phi = \lfloor N_\phi/2 \rfloor$$

for higher orders, use the above correspondence rules

Horndeski theory

Most general scalar-tensor theory with at most 2nd derivatives in EoMs

$$S = \int d^4x \sqrt{-g}\left(\mathcal{L}_2^{\rm H} + \mathcal{L}_3^{\rm H} + \mathcal{L}_3^{\rm H} + \mathcal{L}_4^{\rm H} + \mathcal{L}_5^{\rm H}\right)$$

Horndeski, 1974

$$\mathcal{L}_2^{\rm H} = \Lambda^4 G_2(\phi, X),$$
$$\mathcal{L}_3^{\rm H} = \Lambda G_3(\phi, X)\Box\phi,$$
$$\mathcal{L}_4^{\rm H} = M_P^2 G_4(\phi, X)R + \frac{G_{4,X}}{\Lambda^2}\left((\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right),$$
$$\mathcal{L}_5^{\rm H} = \frac{M_P^2}{\Lambda^3} G_5(\phi, X) G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{G_{5,X}}{6\Lambda^5}\left((\Box\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right),$$

$G_i(\phi, X)$: **generic functions**

$$X \equiv -\partial_\mu\phi\partial^\mu\phi/\left(2\Lambda^4\right)$$

re-discovered as generalized galileon

Deffayet, Deser & Esposito-Farese, 0906.1967

Built upon galileon theory

Nicolis, Rattazzi & Trincherini, 0811.2197

$$\mathcal{L} = \sum_{n=1}^{D} g_n \phi \partial^{\mu_1}\partial_{[\mu_1}\phi\partial^{\mu_2}\partial_{\mu_2}\phi \ldots \partial^{\mu_n}\partial_{\mu_n]}\phi$$

Goldstone mode

$$\phi \to \phi + c + b_\mu x^\mu$$

galileon arise in decoupling limit DGP braneworld, dRGT massive gravity

GW speed constraints on Horndeski theory

First binary neutron star merger

GW170817: both GWs and EMs $\Rightarrow$ $|c_{\rm gw} - 1| \sim 10^{-15}$

multi-messenger astronomy!

Leftover of Horndeski theory after GW speed constraint

if taken as dark energy model

$$\mathcal{L}_2^{\rm H} = \Lambda^4 G_2(\phi, X),$$

$$\mathcal{L}_3^{\rm H} = \Lambda G_3(\phi, X)\Box\phi,$$

$$\mathcal{L}_4^{\rm H} = M_P^2 G_4(\phi, X)R + \frac{G_{4,X}}{\Lambda^2}\left((\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right),$$

$$\mathcal{L}_5^{\rm H} = \frac{M_P^2}{\Lambda^3} G_5(\phi, X) G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{G_{5,X}}{6\Lambda^5}\left((\Box\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right),$$

Creminelli & Vernizzi, 1710.05877
Ezquiaga & Zumalacárregui, 1710.05901
Sakstein & Jain, 1710.05893
Copeland, Kopp, Padilla, Saffin & Skordis, 1810.08239

Caveat for the GW constraints

GWs from LIGO: $10 - 100\ \mathrm{Hz}$

de Rham & Melville, 1806.09417

Cutoff for Horndeksi for dark energy: $\Lambda_3 = (M_P H_0^2)^{\frac{1}{3}} \sim 100\ \mathrm{Hz}$

LIGO GW frequencies ~ EFT cutoff

The GW speed constraints should be taken with a pinch of salt!

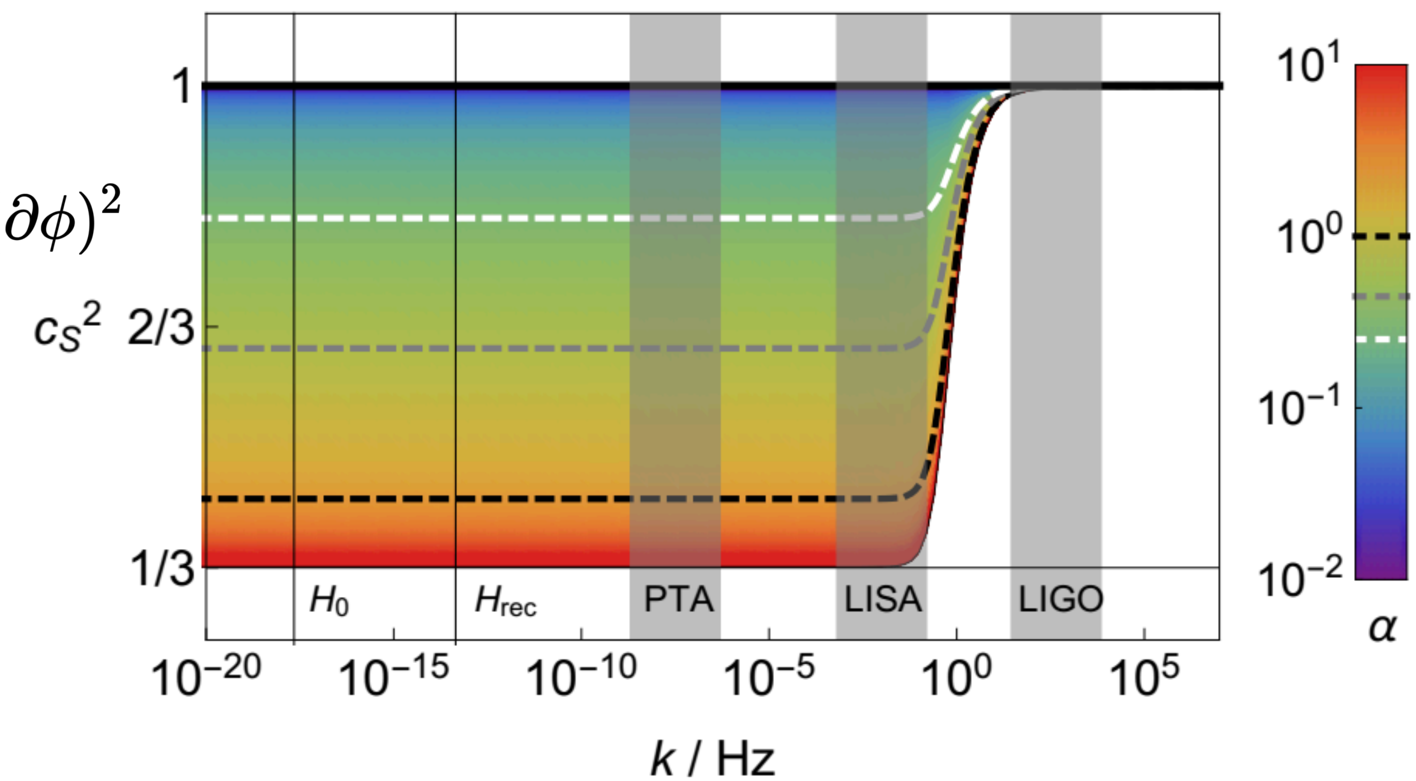
Toy model

$$\mathcal{L}_{\Lambda_*} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}M^2\chi^2 + \frac{\chi}{\Lambda_*}(\partial\phi)^2$$

Integrating out ⇒

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \frac{1}{2\Lambda^4}(\partial\phi)^2\frac{M^2}{M^2-\Box}(\partial\phi)^2$$

$$\omega^2 = k^2 - \frac{4\alpha^2}{1+2\alpha^2}\frac{\omega^2 M^2}{M^2-\omega^2+k^2}$$



LISA will settle it!

Beyond Horndeski theory/DHOST

not 2nd order, yet no Ostragradski ghost

Degenerate Lagrangians

can't fully Legendre transform form $\dot{q}_i$ to p_i

Horndeski theory: degenerate within scalar
beyond Horndeski: degenerate between scalar & metric

Degenerate Higher Order Scalar Tensor (DHOST)

$$\mathcal{L}_1^{\rm d} = \frac{1}{\Lambda^2} A_1(\phi, X) \nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla^\nu \phi$$

$$\mathcal{L}_2^{\rm d} = \frac{1}{\Lambda^2} A_2(\phi, X) (\Box \phi)^2$$

$$\mathcal{L}_3^{\rm d} = \frac{1}{\Lambda^6} A_3(\phi, X) (\Box \phi) \nabla_\mu \phi \nabla^\mu \nabla^\nu \phi \nabla_\nu \phi$$

$$\mathcal{L}_4^{\rm d} = \frac{1}{\Lambda^6} A_4(\phi, X) \nabla_\mu \phi \nabla^\mu \nabla^\rho \phi \nabla_\rho \nabla_\nu \phi \nabla^\nu \phi$$

$$\mathcal{L}_5^{\rm d} = \frac{1}{\Lambda^{10}} A_5(\phi, X) (\nabla^\mu \phi \nabla_\mu \nabla_\nu \phi \nabla^\nu \phi)^2$$

\+ Horndeski terms

subject to:
degenerate conditions

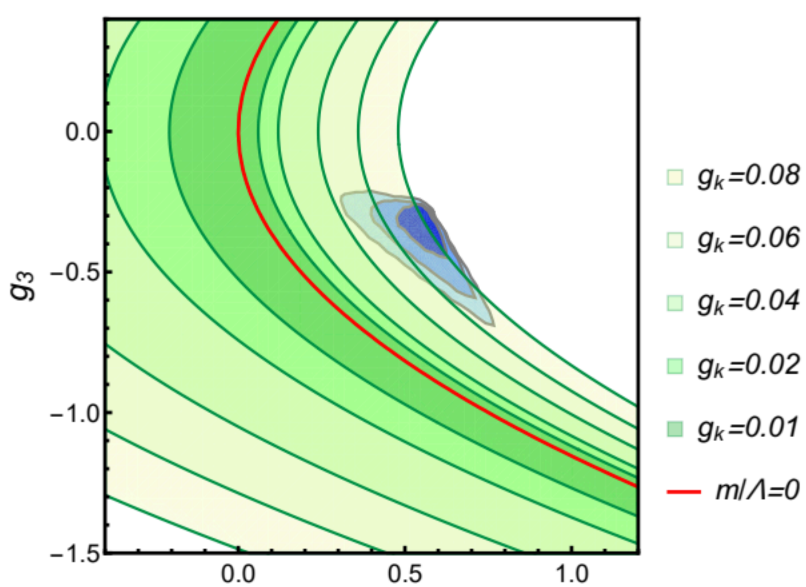
Gleyzes, Langlois, Piazza & Vernizzi, 1404.6495
Deffayet, Deser & Esposito-Farese, 0906.1967
Gao, 1406.0822
……

Positivity bounds on Horndeski and beyond

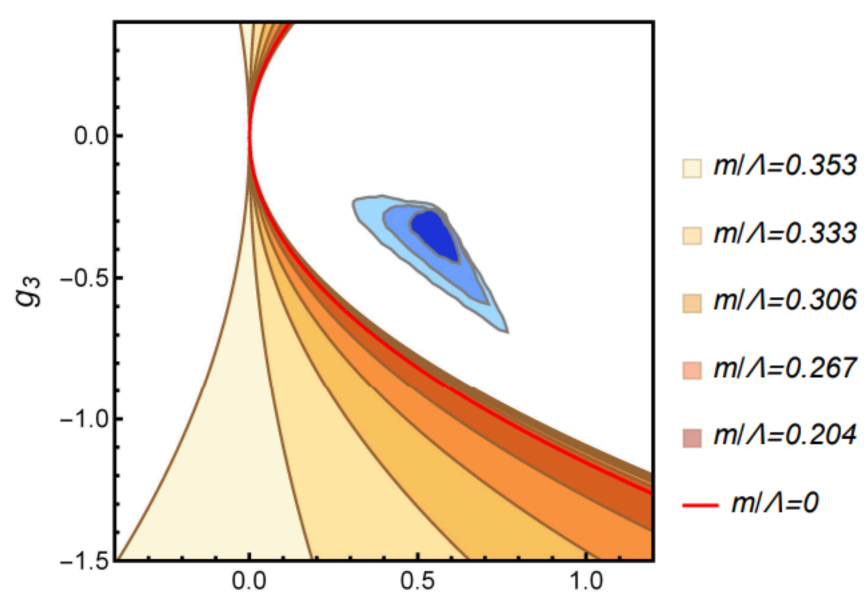
Even positivity bounds on scalar sector already rule out a lot of models!

Positivity bounds + cosmo. data

$(\partial\phi)^4$ deformed galileon

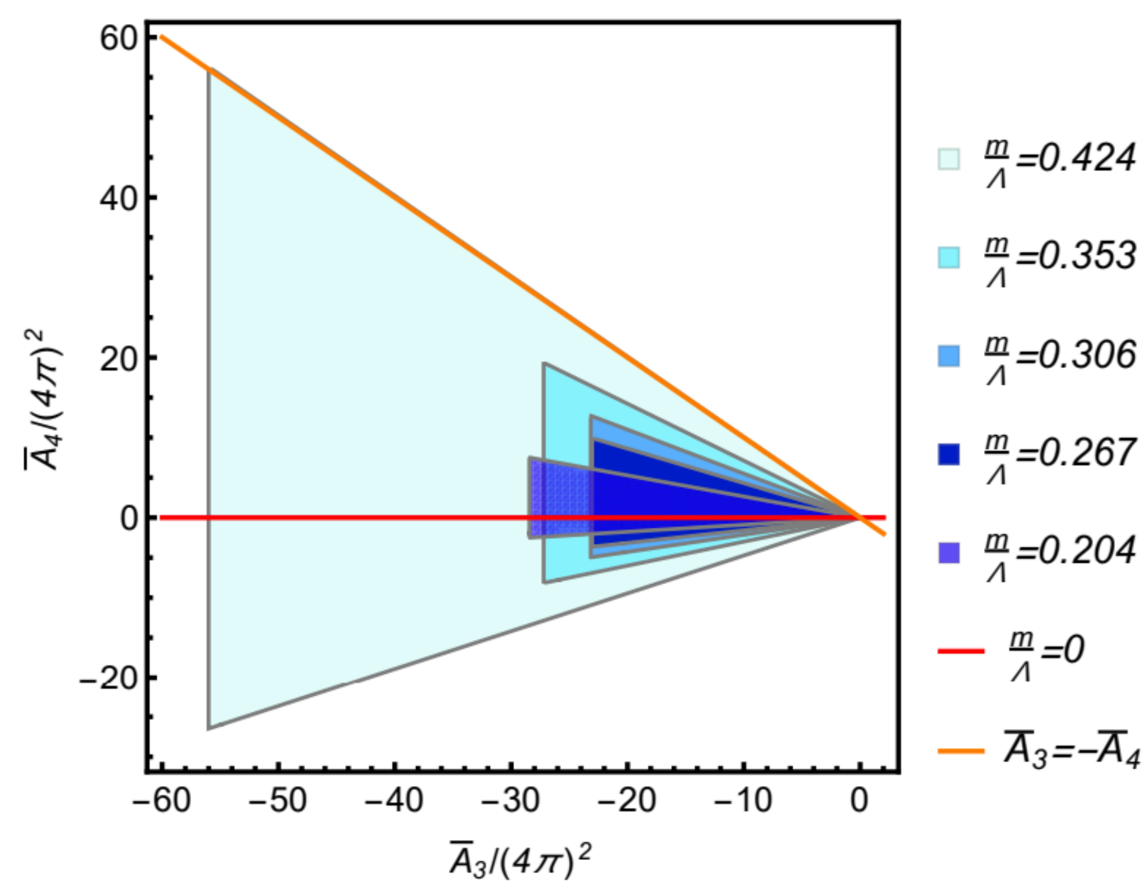


massive galileon



N-I class DHOST

(satisfies GW speed data)



Xu & **SYZ**, 2306.06639

unhealthy hierarchy between m and Λ

EFT of inflation

Weinberg approach $\Delta\mathcal{L} = \sqrt{-g}\left(f_1(\varphi)(\partial\varphi)^2 + f_9(\varphi)C^2_{\mu\nu\rho\sigma} + f_{10}(\varphi)\tilde{C}C\right)$

$M \gg \sqrt{2\epsilon}M_P$

Weinberg, 0804.4291

EFT in broken phase $M^2 \gg \epsilon H M_P$

Cheung, Creminelli, Fitzpatrick, Kaplan & Senatore, 0709.0293

Inflaton background $\phi_0(t)$ choose ϕ as clock $x^i \to x^i + \xi^i(t,\vec{x})$

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{2}M_{\rm Pl}^2 R + M_{\rm Pl}^2\dot{H}g^{00} - M_{\rm Pl}^2\left(3H^2+\dot{H}\right) + \frac{1}{2!}M_2(t)^4\left(g^{00}+1\right)^2 + \frac{1}{3!}M_3(t)^4\left(g^{00}+1\right)^3 + - \frac{\bar{M}_1(t)^3}{2}\left(g^{00}+1\right)\delta K^\mu_{\ \mu} - \frac{\bar{M}_2(t)^2}{2}\delta K^\mu_{\ \mu}{}^2 - \frac{\bar{M}_3(t)^2}{2}\delta K^\mu_{\ \nu}\delta K^\nu_{\ \mu} + \dots\Big].$$

unitarity gauge $\delta\phi = 0$, inflaton eaten by graviton

generic non-Gaussianities
modified sound speed

Goldstone mode (equivalence theorem)

$$S_\pi = \int d^4x\sqrt{-g}\left[M_{\rm Pl}^2\dot{H}(\partial_\mu\pi)^2 + 2M_2^4\left(\dot{\pi}^2 + \dot{\pi}^3 - \dot{\pi}\frac{1}{a^2}(\partial_i\pi)^2\right) - \frac{4}{3}M_3^4\dot{\pi}^3 - \frac{\bar{M}^2}{2}\frac{1}{a^4}\left(\partial_i^2\pi\right)^2 + \dots\right]$$

easy to compute high energy corrections

EFTs of dark energy and black holes

EFT of dark energy (similar to EFT of inflation)

Gubitosi, Piazza & Vernizzi, 1210.0201

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\Big[M_*^2 f R - \rho_D + p_D - M_*^2(5H\dot{f} + \ddot{f}) - \Big(\rho_D + p_D + M_*^2(H\dot{f} - \ddot{f})\Big)g^{00}$$
$$+ M_2^4(\delta g^{00})^2 - \bar{m}_1^3\,\delta g^{00}\delta K - \bar{M}_2^2\,\delta K^2 - \bar{M}_3^2\,\delta K_\mu{}^\nu \delta K^\mu{}_\nu + m_2^2 h^{\mu\nu}\partial_\mu g^{00}\partial_\nu g^{00}$$
$$+ \lambda_1 \delta R^2 + \lambda_2 \delta R_{\mu\nu}\delta R^{\mu\nu} + \mu_1^2 \delta g^{00}\delta R + \gamma_1 C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} + \gamma_2 \epsilon^{\mu\nu\rho\sigma}C_{\mu\nu}{}^{\kappa\lambda}C_{\rho\sigma\kappa\lambda}$$
$$+ \frac{M_3^4}{3}(\delta g^{00})^3 - \bar{m}_2^3\,(\delta g^{00})^2\delta K + \dots\Big],$$

useful for perturbation theory

dark energy and modified gravity in a given FRW background

EFT of hairy black holes

Franciolini, Hui, Penco, Santoni & Trincherini, 1810.07706
Hui, Podo, Santoni & Trincherini, 2111.02072

Hairy background: spherically symmetric & slowly rotating

parametrize perturbations in scalar-tensor theory

Summary

- EFTs are widely in gravitational and cosmological models
    - Einstein EFT
    - Scalar-tensor EFT
- In the theoretical side, a lot of effects have been trying to understand and derive positivity/causality bounds on the Wilson coefficients.
- Positivity bounds are constraints on the IR physics from the UV information. They are robust because they only reply on fundamental principles of S-matrix.
- Positivity bounds on gravitational EFTs are more subtle.

Thank you

Positivity bounds on EFTs

“Anything goes” for EFT coefficients?

EFTs are widely used in modern physics!

Lagrangian
$$\mathscr{L}_{\text{EFT}} = \sum_i \Lambda^4 \, c_i \, \mathscr{O}_i \left(\frac{\text{boson}}{\Lambda}, \frac{\text{fermion}}{\Lambda^{3/2}}, \frac{\partial}{\Lambda} \right)$$

Λ: EFT cutoff c_i: Wilson coefficients or low energy constants

Amplitude
$$\mathscr{A}_{\text{EFT}}(s,t) = \sum_{m,n} \frac{c_{m,n}}{\Lambda^{2m+2n}} s^m t^n$$

Question: **Are Wilson coefficients $c_{m,n}$ allowed to take any values?**

Answer: **No!**

Positivity/Causality bounds

high energy UV theory

maybe unknown, but assume causality, unitarity, ...

↓

Positivity bounds/
Causality bounds

S-matrix / EFT bootstrap

low energy EFT

constraints on Wilson coefficients

Simple example

Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi, 2006

$$\mathscr{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\lambda}{\Lambda^4}(\partial_\mu\phi\partial^\mu\phi)^2 + \cdots$$

$$A(s, t=0) = \cdots + \frac{2\lambda s^2}{\Lambda^4} + \cdots$$

"First" positivity bound: $\lambda > 0$

$$\mathscr{L}_{\mathrm{DBI}} \sim -\sqrt{1+(\partial\phi)^2}$$ ✓

$$\mathscr{L}_{\overline{\mathrm{DBI}}} \sim \sqrt{1-(\partial\phi)^2}$$ ✗

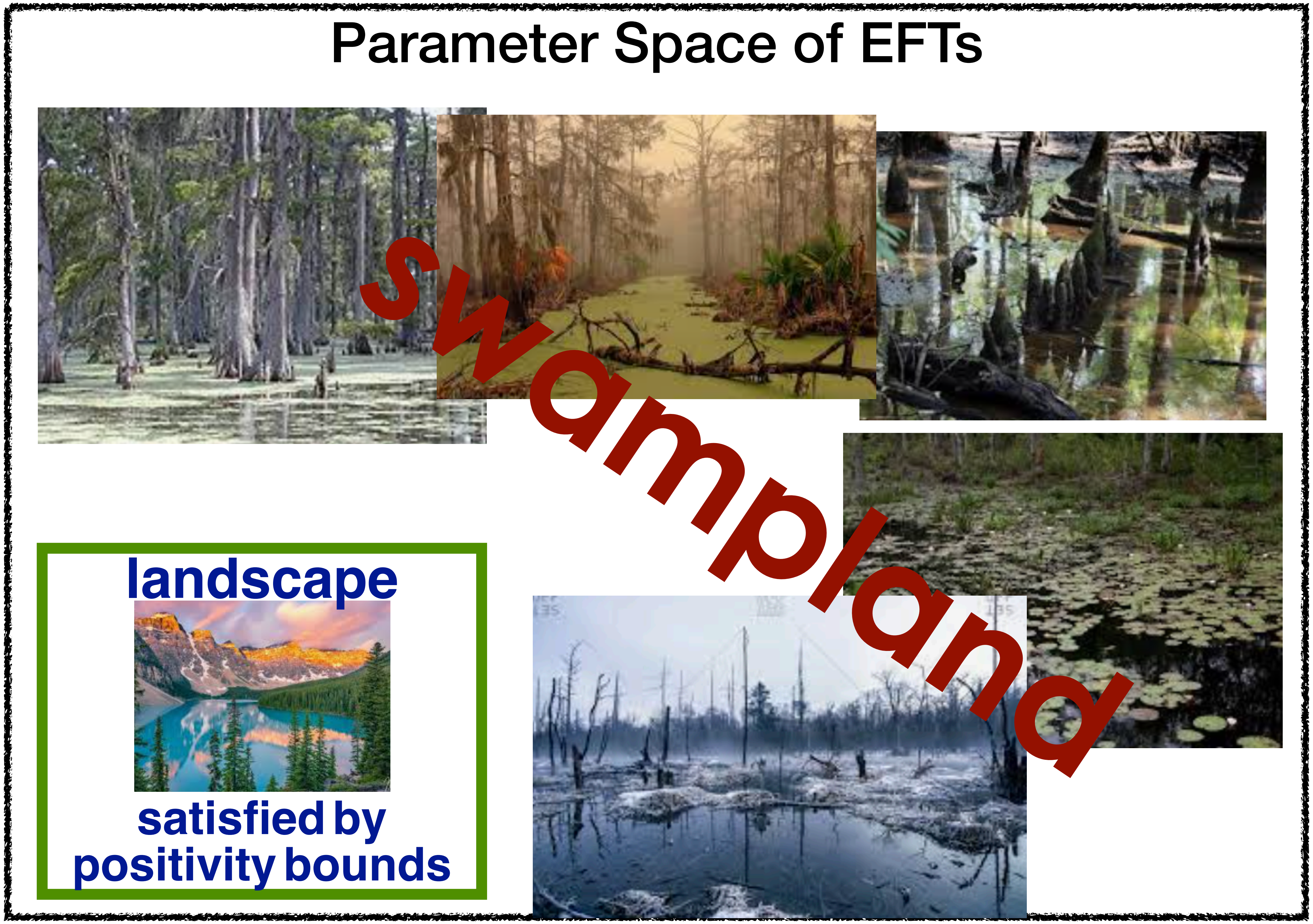
Significant advances since 2017:

Adams, Alberte, Aoki, Arkani-Hamed, Baumann, Bellazzini, Bern, Caron-Huot, Chandrasekaran, Cheung, Chiang, Creminelli, de Rham, Dubovsky, Elias Miro, Fuks, Grall, Green, Guerrieri, Hanada, Henriksson, Herrero-Valea, Hirano, Huang, Jaitly, Janssena, Jenkins, Kim, Kundu, Lee, Lewandowski, Li, Liu, McPeak, Melville, Momeni, Noller, Noumi, Nicolis, O'Connell, Penedones, Porto, Rattazzi, Remmen, Riembau, Riva, Rodd, Rodina, Russo, Rumbutis, Santos-Garcia, Senatore, Serra, Sgarlata, Shahbazi-Moghaddam, Shiu, Sinha, Timiryasov, Tokareva, Tokuda, Tolley, Trincherini, Trott, Van Duong, Vichi, Wang, Weng, Xu, Yamashita, Yang, Yao, Zahed, Zhang, Zhiboedov, Zhou, …

Snowmass White Paper: UV Constraints on IR Physics, de Rham, Kundu, Reece, Tolley & **SYZ**, 2203.06805

Similar to swampland idea

But positivity bounds take more conservative approach



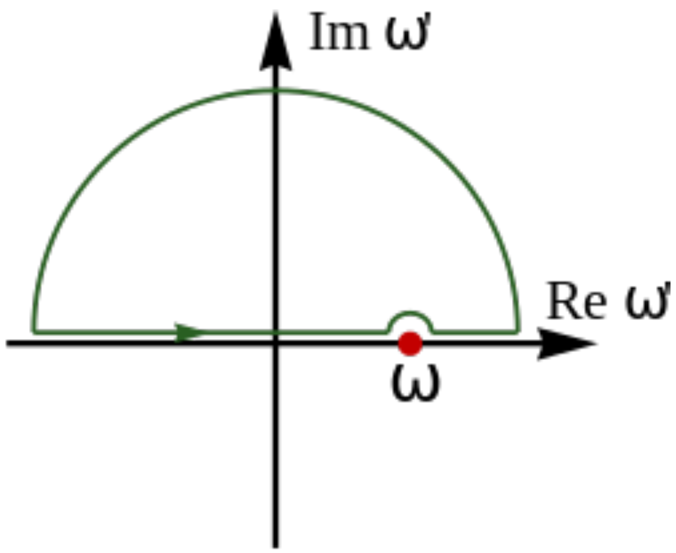
Causality implies analyticity

Kramers-Kronig dispersion relation

$$f(t < 0) = 0$$

$\tilde{f}(\omega)$ square-integrable

→ Titchmarsh's theorem: $\tilde{f}(\omega)$ analytic in upper ω plane

→ 

→ $$\tilde{f}(\omega) = \frac{1}{i\pi}\mathcal{P}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega'}{\omega' - \omega}\tilde{f}(\omega')$$

eg, complex refractive index n

Relativistic version: response restricted with light-cone

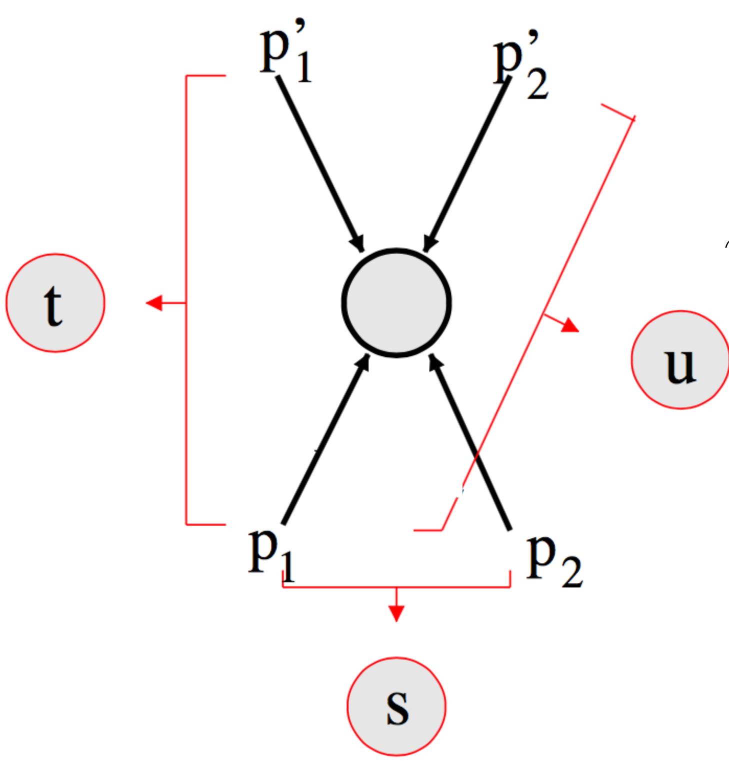
$$f(t, \boldsymbol{x}) = \theta(t - \boldsymbol{\xi}\cdot\boldsymbol{x})f(t, \boldsymbol{x})$$
$$\boldsymbol{\xi}^2 < 1$$

→ $$\tilde{f}(\omega, \boldsymbol{k}_0 + \omega\boldsymbol{\xi}) = \frac{1}{i\pi}\mathcal{P}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega'}{\omega' - \omega}\tilde{f}(\omega', \boldsymbol{k}_0 + \omega'\boldsymbol{\xi})$$

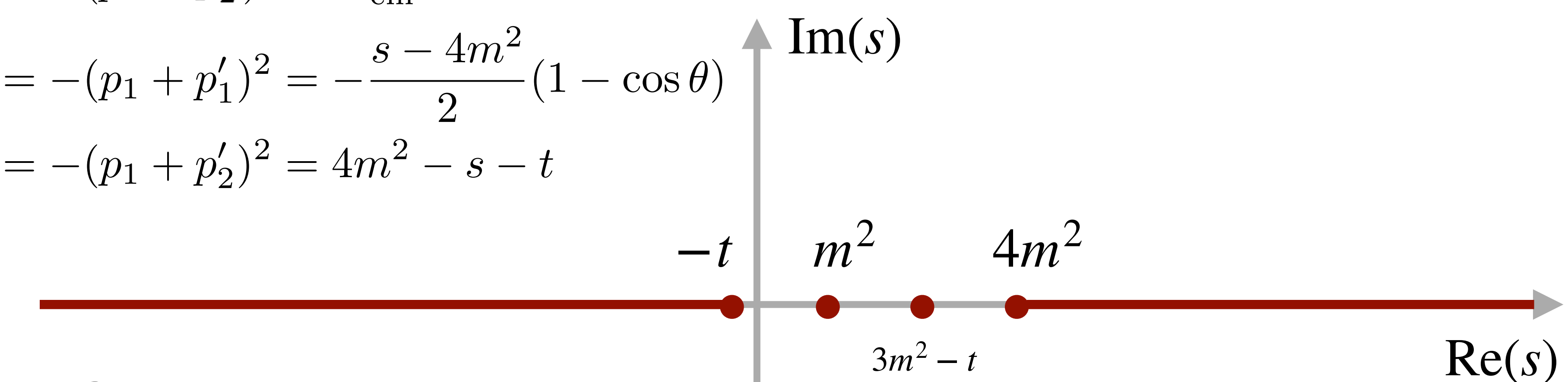
Analyticity of scattering amplitude

$A(s,t)$ as analytic function

S-matrix program in 60'
Mandelstam, Martin, …



$$s = -(p_1 + p_2')^2 = E_{\rm cm}^2$$
$$t = -(p_1 + p_1')^2 = -\frac{s - 4m^2}{2}(1 - \cos\theta)$$
$$u = -(p_1 + p_2')^2 = 4m^2 - s - t$$



Crossing symmetry

$$A(s,t) = A(u,t) = A(t,s)$$

Locality: $A(s,t)$ is polynomially bounded at high energies

Froissart(-Martin) bound: Froissart, 1961; Martin, 1962

$$\lim_{s\to\infty} |A(s,t)| < C s^{1+\epsilon(t)}, \quad t < 4m^2, \quad 0 < \epsilon(t) < 1$$

Unitarity

Unitarity: conservation of probabilities $S^\dagger S = 1 \Rightarrow T - T^\dagger = iT^\dagger T$

Generalized optical theorem

$$A(I \to F) - A^*(F \to I) = i \sum_X \int d\Pi_X (2\pi)^4 \delta^4(p_I - p_X) A(I \to X) A^*(F \to X)$$

optical theorem ($\theta = 0$): $\mathrm{Im}[A(I \to I)] \sim \sum_X \sigma(I \to X) > 0$

Partial wave expansion: $A(s,t) \sim \sum_{\ell=0}^{\infty} (2\ell + 1) P_\ell(\cos\theta) a_\ell(s)$

(2-2 scattering, for scalar)

Partial wave unitary bounds:

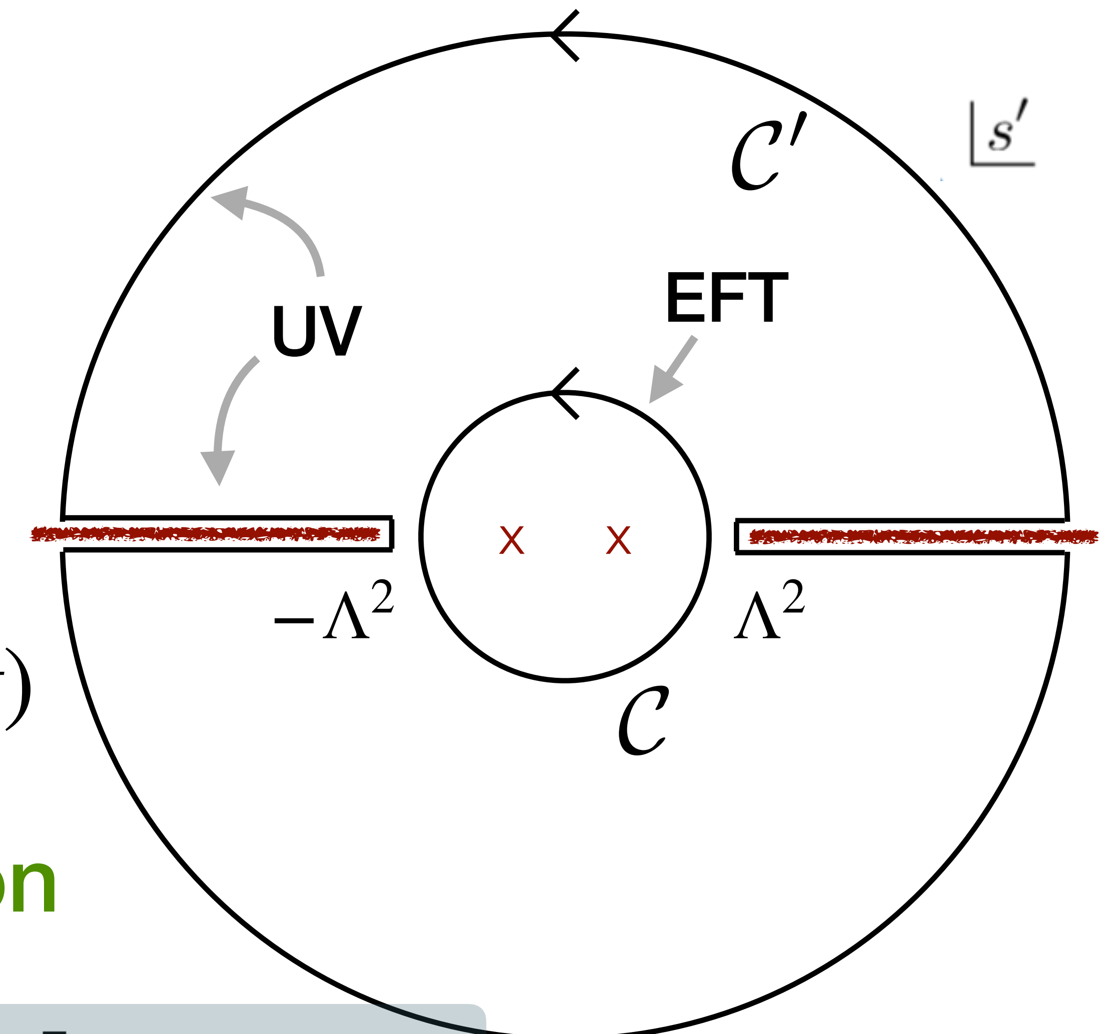
$$0 \leq |a_\ell(s)|^2 \leq \mathrm{Im}\, a_\ell(s) \leq 1$$

Fixed t dispersion relation

- Analyticity in complex s plane (fixed t)

$$A(s,t) = \frac{1}{2\pi i}\oint_{\mathcal{C}} ds' \, \frac{A(s',t)}{s'-s}$$

- Froissart bound $|A(s' \to \infty, t)| < s'^{2-\epsilon}$
- su crossing symmetry $A(s,t) = A(u,t)$



Twice subtracted dispersion relation

$$A(s,t) \sim \int_{\Lambda^2}^{\infty} \frac{\mathrm{d}\mu}{\pi\mu^2}\left[\frac{s^2}{\mu - s} + \frac{u^2}{\mu - u}\right] \mathrm{Im}\, A(\mu,t)$$

EFT amplitude **IR/UV connection** **UV full amplitude**

Forward positivity bounds

Forward limit $t = 0$

$$A(s,0) \sim \int_{\Lambda^2}^{\infty} \frac{\mathrm{d}\mu}{\pi\mu^2}\left[\frac{s^2}{\mu - s} + \frac{s^2}{\mu + s}\right] \mathrm{Im}\, A(\mu, 0)$$

$$c_{2,0}s^2 + c_{4,0}s^4 + \cdots = \left(\int \frac{2\,\mathrm{d}\mu}{\pi\mu^3}\mathrm{Im}\, A(\mu,0)\right)s^2 + \left(\int \frac{2\,\mathrm{d}\mu}{\pi\mu^5}\mathrm{Im}\, A(\mu,0)\right)s^4 + \cdots$$

matching → **Sum rules:**

$$c_{2n,0} = \int \frac{2\,\mathrm{d}\mu}{\pi\mu^{1+2n}}\mathrm{Im}\, A(\mu,0)$$

μ: scale of UV particles

"First" bounds

Optical theorem
$\mathrm{Im}[A(s,0)] \propto \sigma(s) > 0$ → $c_{2n,0} > 0$

Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi, 2006 + earlier works

Recent developments of positivity bounds

Generalization away from $t = 0$

$$A(s,t) \sim \int_{\Lambda^2}^{\infty} \frac{\mathrm{d}\mu}{\pi\mu^2} \left[\frac{s^2}{\mu - s} + \frac{u^2}{\mu - u} \right] \mathrm{Im}\, A(\mu, t)$$

partial wave unitarity

positivity of Legendre polynomial

$$\frac{\partial^n}{\partial t^n} \mathrm{Im}[A(s,t)] > 0, \quad s \geq 4m^2, \; 0 \leq t < 4m^2$$

Recurrent Y bounds:

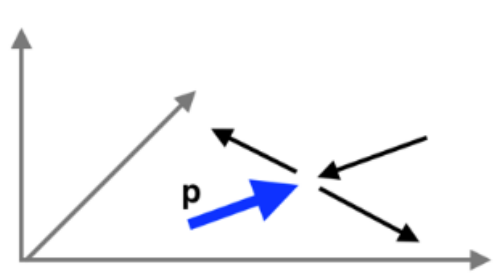
de Rham, Melville, Tolley & **SYZ**,1702.06134

$$Y^{(2N,M)} = \sum_{r=0}^{M/2} c_r B^{(2N+2r, M-2r)} + \frac{1}{\mathcal{M}^2} \sum_{k \text{ even}}^{(M-1)/2} (2(N+k)+1) \beta_k Y^{(2(N+k), M-2k-1)} > 0$$

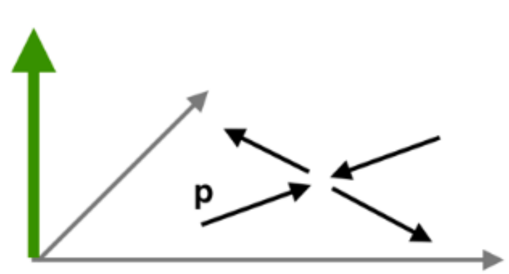
Valid for massive particles with spin
if use transversity formalism

de Rham, Melville, Tolley & **SYZ**,1706.02712

Helicity



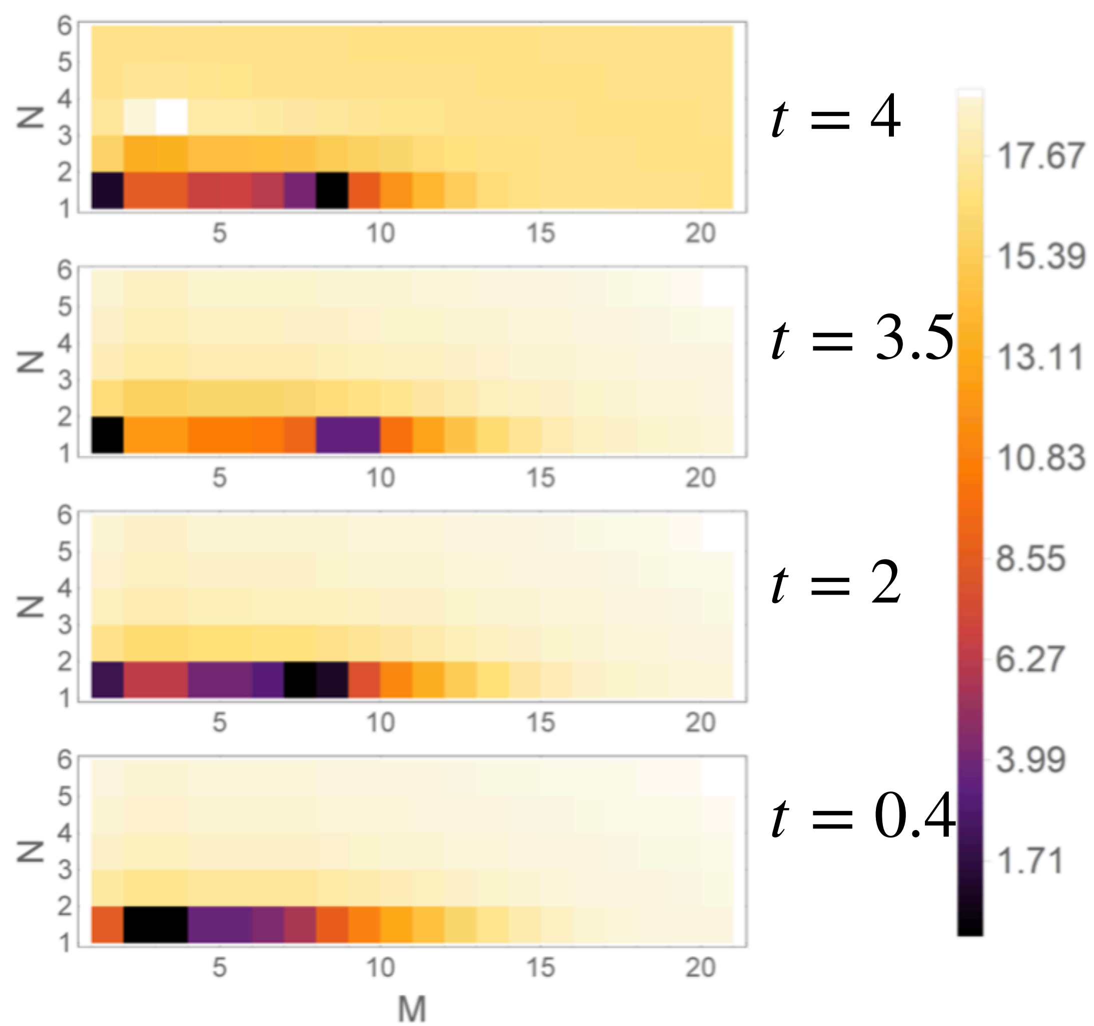
Transversity



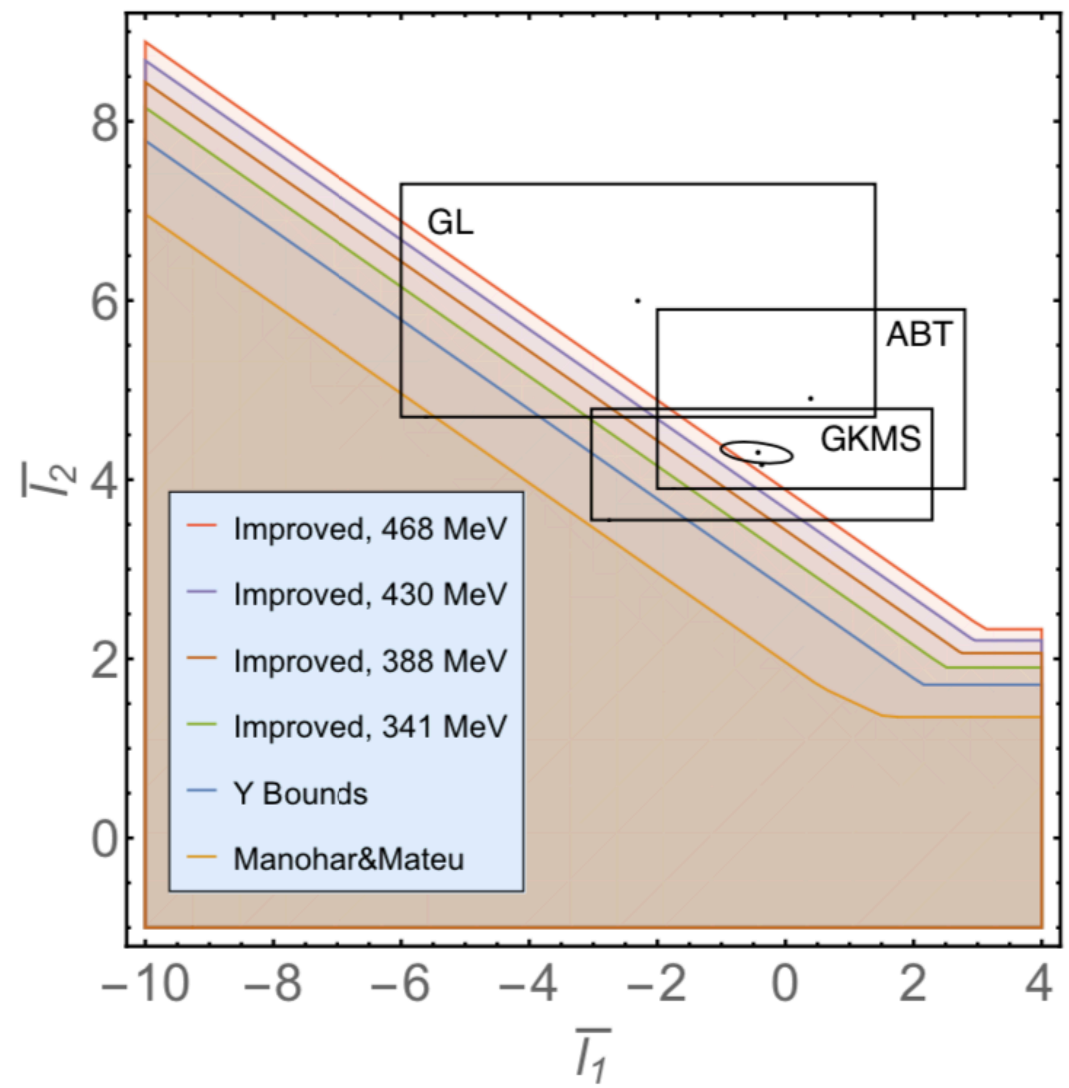
Y bounds on SU(2) ChPT

Wang, Feng, Zhang & **SYZ,** 2004.03992

best bounds are away from $t = 0$



$Y^{(2N,M)}(t)$



bounds on $O(p^4)$ coefficients

EFT-Hedron for $t = 0$

Arkani-Hamed, Huang & Huang, *talks in 2017*, 2012.15849
Bellazzini, Miro, Rattazzi, Riembau & Riva, 2011.00037

$$c_{2n,0} = \int_{\Lambda^2}^{\infty} \frac{2\,\mathrm{d}\mu}{\pi \mu^{1+2n}} \mathrm{Im}\, A(\mu, 0) \quad \xrightarrow{x \equiv \Lambda/\mu} \quad c_{2n,0} = \int_0^1 x^n d\rho(x)$$

This is a Hausdorff moment problem!

Solution:

Define Hankel matrix $H(c_{2n,0}) = \begin{pmatrix} c_{2,0} & c_{4,0} & c_{6,0} & \\ c_{4,0} & c_{6,0} & c_{8,0} & \cdots \\ c_{6,0} & c_{8,0} & c_{10,0} & \\ & \vdots & & \ddots \end{pmatrix}$

nonlinear positivity bounds

$$H(c_{2n,0}) \succeq 0 \quad \& \quad H^{\mathrm{shift}}(c_{2n,0}) \equiv H(c_{2n,0})\Big|_{c_{2n,0} \to c_{2n+2,0}} \succeq 0$$

$$\& \quad H(c_{2n,0}) - H^{\mathrm{shift}}(c_{2n,0}) \succeq 0$$

Why su bounds typically one-sided?

$$\sum_{i,j} c_{i,j} s^i t^j = A(s,t) \sim \int_{\Lambda^2}^{\infty} \frac{\mathrm{d}\mu}{\pi\mu^2} \left[\frac{s^2}{\mu - s} + \frac{u^2}{\mu - u} \right] \mathrm{Im}\, A(\mu, t)$$

Expand dispersion relation and match $s^i t^j$ on both sides

partial wave expansion: $A(s,t) \sim \Sigma_\ell P_\ell(1 + 2t/s) a_\ell(s)$

unitarity: $0 \leq |a_\ell(s)|^2 \leq \mathrm{Im}\, a_\ell(s) \leq 1$

Sum rules for $t \neq 0$:

$$c_{i,j} \sim \sum_\ell \int d\tilde{\mu} \frac{D_{i,j}(\eta)}{\mu^{i+j}}$$

$$d\tilde{\mu} \equiv d\mu\, \mathrm{Im} a_\ell(s)$$

$$\eta \equiv \ell(\ell + 1)$$

$D_{i,j}$ is polynomial of η that is bounded below

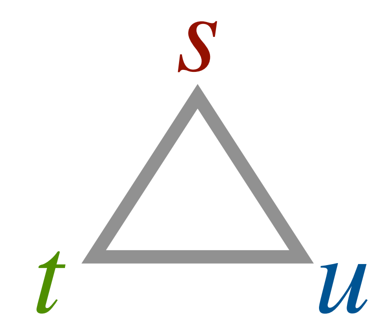
Lower bounds

$$c_{i,j} \sim \sum_\ell \int d\tilde{\mu} \frac{D_{i,j}(\eta)}{\mu^{i+j}} > D_{i,j}^{\min} \sum_\ell \int d\tilde{\mu} \frac{1}{\mu^{i+j}} = D_{i,j}^{\min} c_{2,0}$$

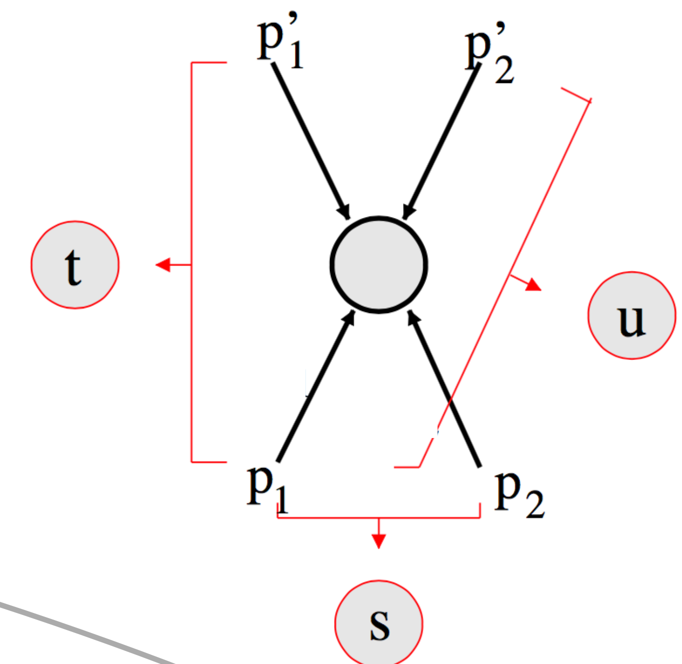
Tolley, Wang & **SYZ**, 2011.02400

$$D_{i,j}^{\min} = \min_\eta \left[D_{i,j}(\eta) \right]$$

Magic of crossing symmetry



We have not used *st* symmetry



$$A(u,t) = A(s,t) = A(t,s)$$

$$\int_{\Lambda^2}^{\infty} \frac{d\mu}{\pi\mu^2}\left[\frac{s^2}{\mu - s} + \frac{u^2}{\mu - u}\right] \mathrm{Im}\, A(\mu, t) \sim \int_{\Lambda^2}^{\infty} \frac{d\mu}{\pi\mu^2}\left[\frac{t^2}{\mu - t} + \frac{u^2}{\mu - u}\right] \mathrm{Im}\, A(\mu, s)$$

Null constraints

Tolley, Wang & **SYZ**, 2011.02400
Caron-Huot & Duong, 2011.02957

$$\sum_{\ell} \int d\mu \frac{\mathrm{Im} a_\ell(\mu)}{\mu^{i+j}} \Gamma_{i,j}^{(n)}(\ell) = 0$$

st crossing imposes constraints on $\mathrm{Im} a_\ell$

Powerful two-sided bounds

Add null constraints to sum rules:

$$\sum_\ell \int d\tilde{\mu} \frac{\Gamma^{(n)}_{i,j}(\eta)}{\mu^{i+j}} = 0$$

$$c_{i,j} \sim \sum_\ell \int d\tilde{\mu} \frac{D_{i,j}(\eta) + \sum_n \alpha_n \Gamma^{(n)}_{i,j}(\eta)}{\mu^{i+j}}$$

can choose α_n to make $D_{i,j} + \Sigma_n \alpha_n \Gamma^{(n)}_{i,j}$ bounded from blow and above

α_n can be positive or negative

before: $D_{i,j}$ only has **min**

now: $D_{i,j} + \Sigma_n \alpha_n \Gamma^{(n)}_{i,j}$ can have **min and max**

→ **Wilson coeff's $c_{i,j}$ have two-sided bounds**

Tolley, Wang & **SYZ**, 2011.02400

Can further add different order $\Gamma^{(n)}_{i,j}(\eta)$

optimize via linear programing

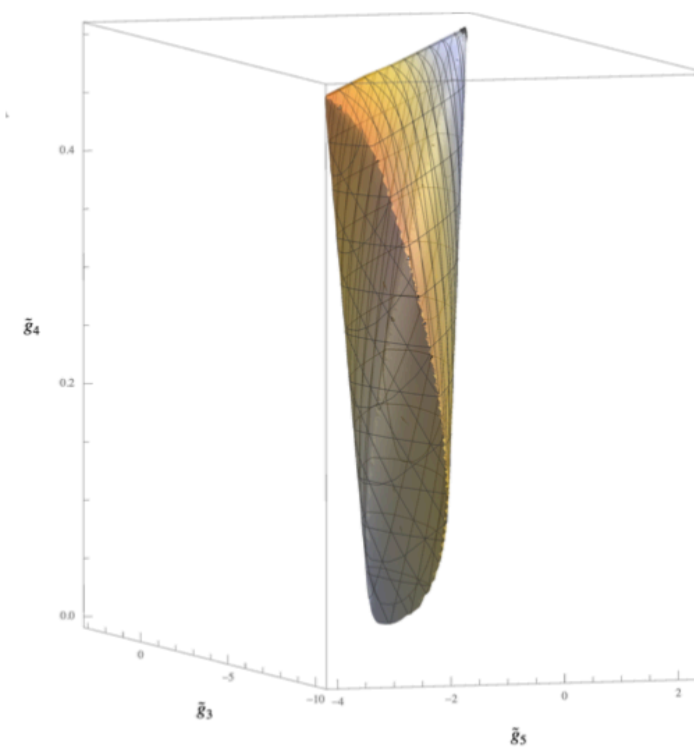
Caron-Huot & Duong, 2011.02957

Two-sided bounds

$$A(s,t) \sim c_{2,0}s^2 + c_{2,1}s^2t + c_{2,2}s^2t^2 + \cdots$$

All Wilson coefficients are parametrically $\lesssim O(1)$!

(m,n)	Lower bounds	Upper bounds
$(1,1)$	$c_{1,1} > -\frac{3}{2}\sqrt{c_{1,0}c_{2,0}}$	$c_{1,1} < 8\sqrt{c_{1,0}c_{2,0}}$
$(2,1)$	$c_{2,1} > -\frac{5}{2}\sqrt{c_{2,0}c_{3,0}}$	$c_{2,1} < \frac{465}{38}\sqrt{c_{2,0}c_{3,0}}$
$(2,2)$	$c_{2,2} > -\frac{9}{2}c_{3,0}$	$c_{2,2} < \frac{2961}{58}c_{3,0}$
$(3,1)$	$c_{3,1} > -\frac{7}{2}\sqrt{c_{3,0}c_{4,0}}$	$c_{3,1} < \frac{1097}{58}\sqrt{c_{3,0}c_{4,0}}$
$(3,2)$	$c_{3,2} > -7c_{4,0}$	$c_{3,2} < \frac{10027}{59}c_{4,0}$
$(3,3)$	$c_{3,3} + \frac{3}{4}c_{4,1} > -\frac{147}{8}\sqrt{c_{4,0}c_{5,0}}$, $c_{3,3} - 8c_{4,1} > -154\sqrt{c_{4,0}c_{5,0}}$, $c_{3,3} - \frac{481}{12}c_{4,1} > -\frac{7777}{8}\sqrt{c_{4,0}c_{5,0}}$, $c_{3,3} - 104c_{4,1} > -3369\sqrt{c_{4,0}c_{5,0}}$	$c_{3,3} - \frac{650}{41}c_{4,1} < -\frac{2310}{41}\sqrt{c_{4,0}c_{5,0}}$
$(4,2)$	$c_{4,2} > -\frac{17}{2}c_{5,0}$	$c_{4,2} < \frac{3923}{12}c_{5,0}$
$(4,3)$	$c_{4,3} + \frac{3}{4}c_{5,1} > -\frac{253}{8}\sqrt{c_{5,0}c_{6,0}}$, $c_{4,3} - \frac{180}{41}c_{5,1} > -\frac{8705}{82}\sqrt{c_{5,0}c_{6,0}}$, $c_{4,3} - \frac{325}{12}c_{5,1} > -\frac{16825}{24}\sqrt{c_{5,0}c_{6,0}}$, $c_{4,3} - \frac{169}{2}c_{5,1} > -\frac{11187}{4}\sqrt{c_{5,0}c_{6,0}}$ $c_{4,3} - \frac{743}{4}c_{5,1} > -\frac{63279}{8}\sqrt{c_{5,0}c_{6,0}}$	$c_{4,3} - \frac{73153}{1748}c_{5,1} < -\frac{708543}{3496}\sqrt{c_{5,0}c_{6,0}}$
$(4,4)$	$c_{4,4} + \frac{25}{24}c_{5,2} > -\frac{147}{8}c_{6,0}$, $c_{4,4} - \frac{125}{37}c_{5,2} > -\frac{71175}{74}c_{6,0}$, $c_{4,4} - \frac{785}{52}c_{5,2} > -\frac{83490}{13}c_{6,0}$, $c_{4,4} - \frac{2485}{69}c_{5,2} > -\frac{1144125}{46}c_{6,0}$	$c_{4,4} - 15c_{5,2} < -\frac{195}{2}c_{6,0}$, $c_{4,4} + \frac{368085}{36544}c_{5,2} < -\frac{2365845}{18272}c_{6,0}$



tree-level in EFT

used to be a folklore, called "naturalness/dimensional analysis"
but now a rigorous QFT theorem

Tolley, Wang & **SYZ**, 2011.02400
Caron-Huot & Duong, 2011.02957

Ruling out Galileon

$$\pi \to \pi + c + b_\mu x^\mu, \quad c, b_\mu = const$$

- linked to dRGT massive gravity, DGP braneworld
- applications in cosmology

original Galileon marginally ruled out by

Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi, 2006

Weakly broken Galileon theories $\mathcal{L} \sim \mathcal{L}_{\text{galileon}} - \frac{m^2}{2}\pi^2$

↓ *stu* symmetric bounds

Tolley, Wang & **SYZ**, 2011.02400

$\Lambda \sim m$ **not a valid EFT**

also inconsistent with observational constraints

Xu & **SYZ**, 2306.XXXXX

Alternative methods

Fully crossing symmetric dispersion relation

$$\mathcal{M}_0(s_1,s_2)=\alpha_0+\frac{1}{\pi}\int_{\frac{2\mu}{3}}^{\infty}\frac{ds_1'}{s_1'}\mathcal{A}\Big(s_1';s_2^{(+)}(s_1',a)\Big)\times H(s_1';s_1,s_2,s_3)$$

Sinha & Zahed, PRL, 2012.04877 + locality constraints; connection to geometric function theory

Locality constraints analytically solved Song, 2305.03669

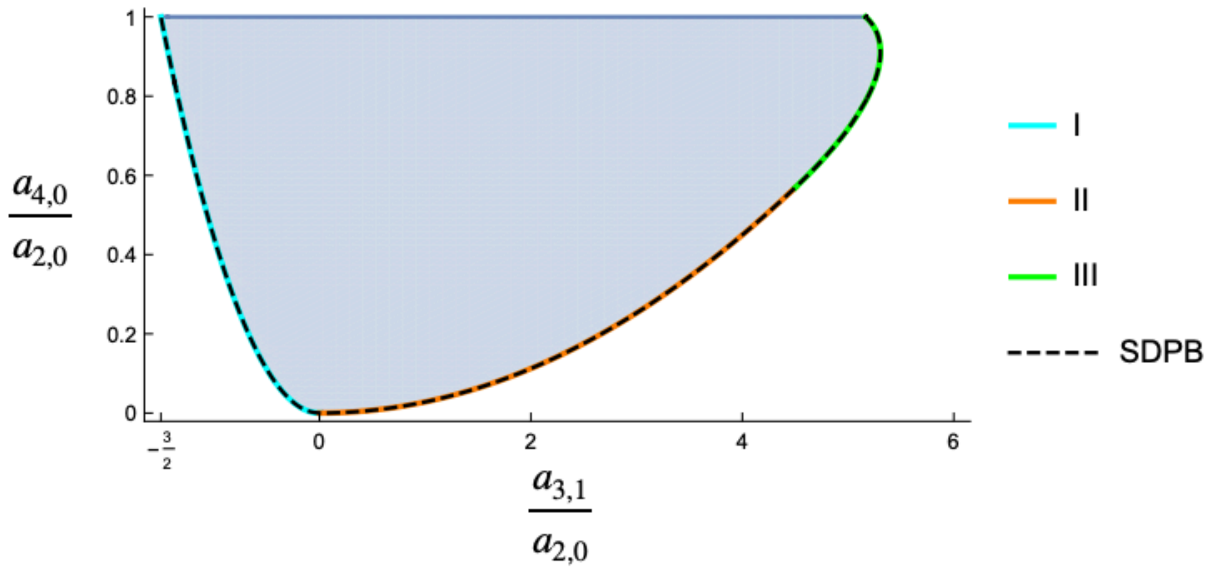
$$\mathcal{M}^{(s)}(\mathbf{s})=\alpha_0+\frac{1}{\pi}\int\frac{d\sigma}{\sigma}\left(\frac{(2\sigma^3+s_1s_2s_3)\mathcal{A}\left(\sigma,s_\pm\left(\sigma,-s_1s_2s_3/\sigma^3\right)\right)}{(\sigma-s_1)(\sigma-s_2)(\sigma-s_3)}-2\mathcal{A}(\sigma,0)\right)$$

stu EFTHedron

reduce to bi-variate moment problem
(GL rotations + triple-crossing slices)

mostly analytical method

Chiang, Huang, Li, Rodina & Weng, 2105.02862



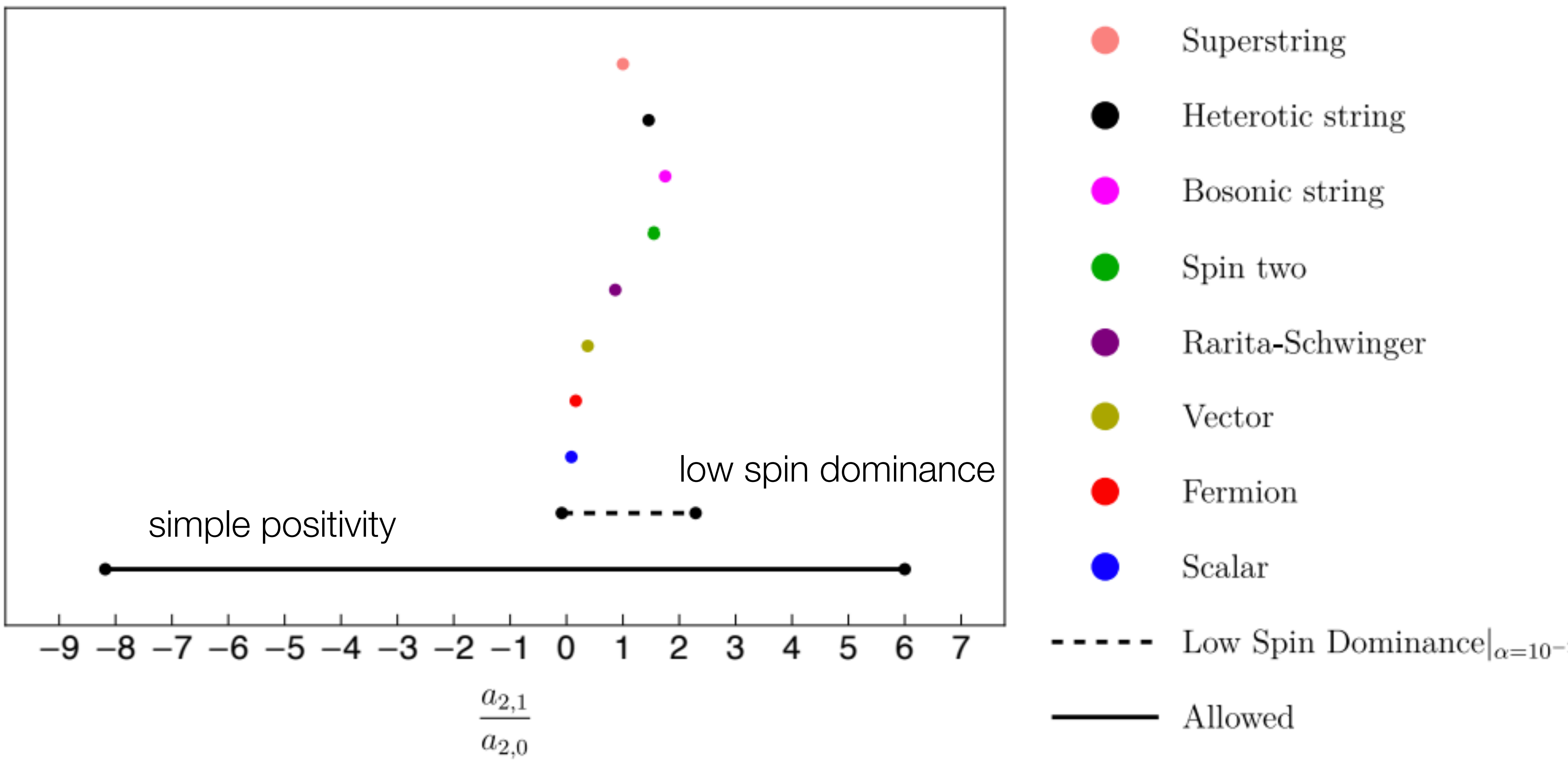
Bounds on gravitational EFT

Bern, Kosmopoulosa & Zhiboedov, 2103.12728

Known theories $\ll$ positivity ?

tree level string amplitude

1-loop $hh \to hh$ amplitude with heavy matter



Multi-field generalization

Partial wave unitary (for identical particle)

$$\mathrm{Im}\, a_\ell^{iiii} = \sum_X a_\ell^{ii\to X} \left(a_\ell^{ii\to X}\right)^* = \sum_X |a_\ell^{ii\to X}|^2 > 0$$

use **linear programing** to obtain optimal bounds

Generalized optical theorem (for multiple fields)

$$\mathrm{Im}\, a_\ell^{ijkl} = \sum_X a_\ell^{ij\to X} \left(a_\ell^{kl\to X}\right)^* \succeq 0$$

use **semi-definite programing** to obtain optimal bounds

SDP with a continuous decision variable

μ: the UV scale; solvable by SDPB

Du, Zhang & **SYZ**, 2111.01169

Bi-scalar theory

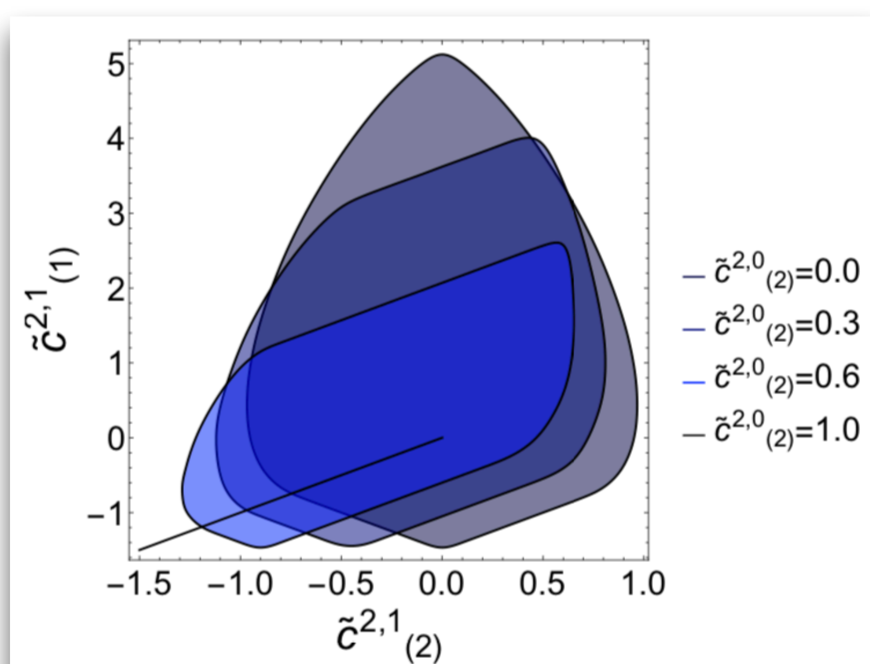
Sum rules

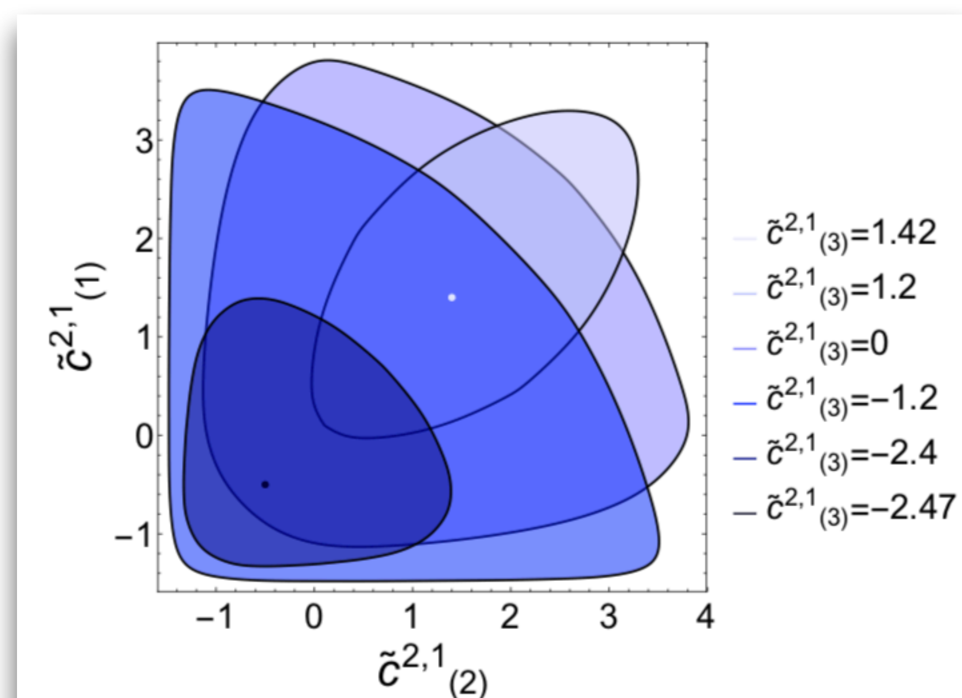
$$c^{m,n}_{ijkl} = \left\langle C^{m,n}_{ijkl} \right\rangle \equiv \left\langle \left[m^{ij}_\ell m^{kl}_\ell + (-1)^m m^{il}_\ell m^{kj}_\ell \right] \frac{C^{m,n}_\ell}{\mu^{m+n+1}} \right\rangle \qquad m^{ij}_\ell \sim \mathrm{Im}\, a^{ij\to X}_\ell$$

and more null constraints

Z_2 bi-scalar theory

Du, Zhang & **SYZ**, 2111.01169





Graviton t-channel pole

Spin-2 pole s^2/t survives twice subtraction

$$\frac{1}{M_P^2 t} + (\cdots) \sim \int_{\Lambda^2}^{\infty} \frac{\mathrm{d}\mu}{\mu^3} \mathrm{Im}\, A(\mu, t)(\cdots)$$

Bounds are not strictly positive

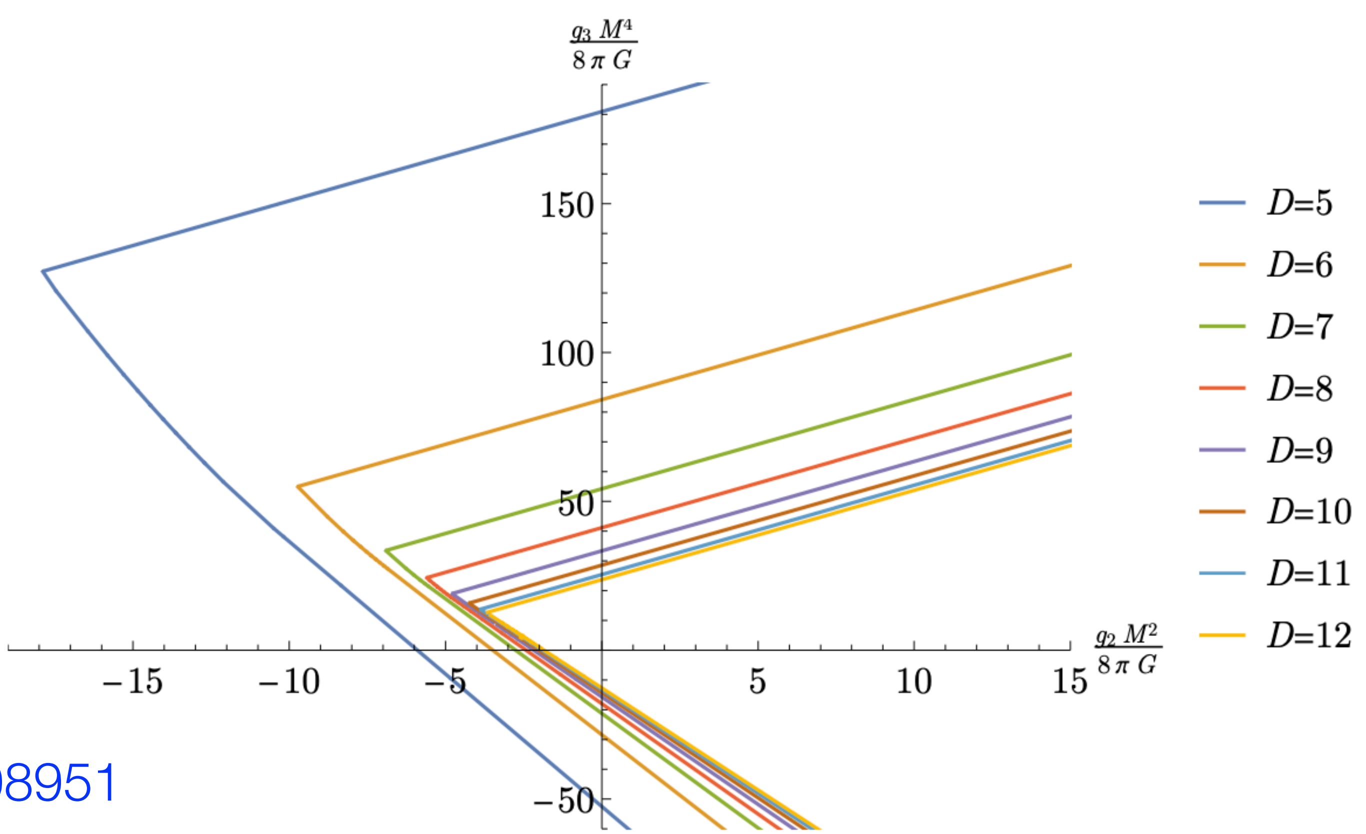
Alberte, de Rham, Jaitly & Tolley, 2007.12667
Tokuda, Aoki & Hirano, 2007.15009

$$a_{2,0} > -\frac{\Lambda^2}{M_{\mathrm{Pl}}^2} \times \mathcal{O}(1)$$

Numerical bounds

functional optimization

impact parameter $b = \ell/\mu$

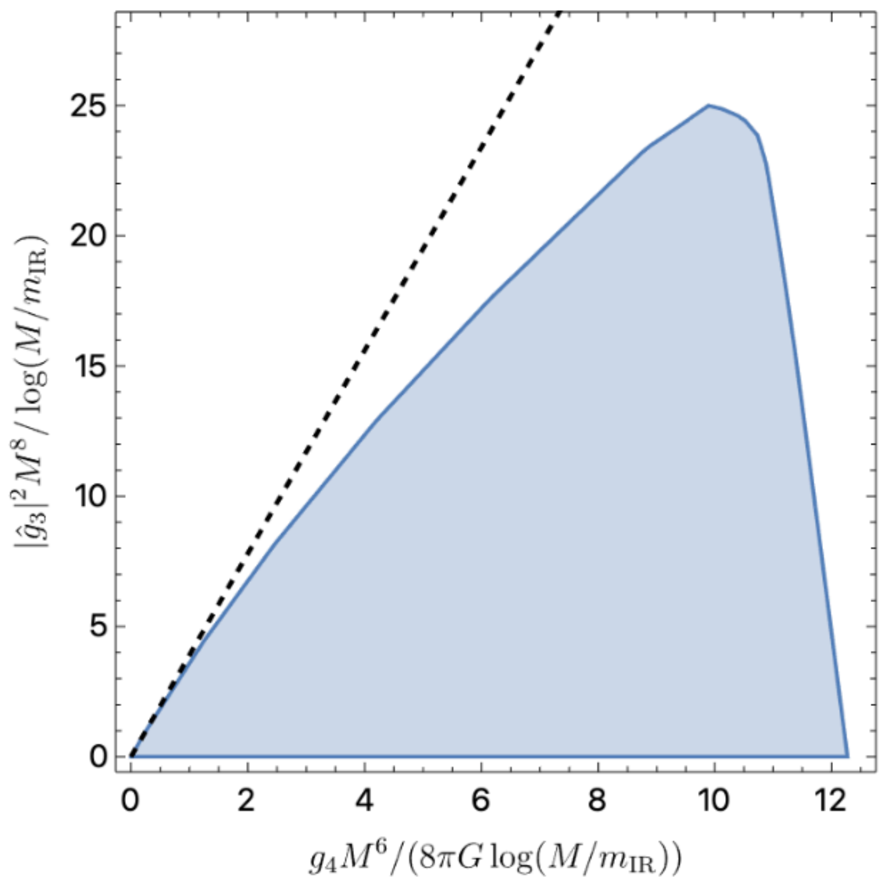


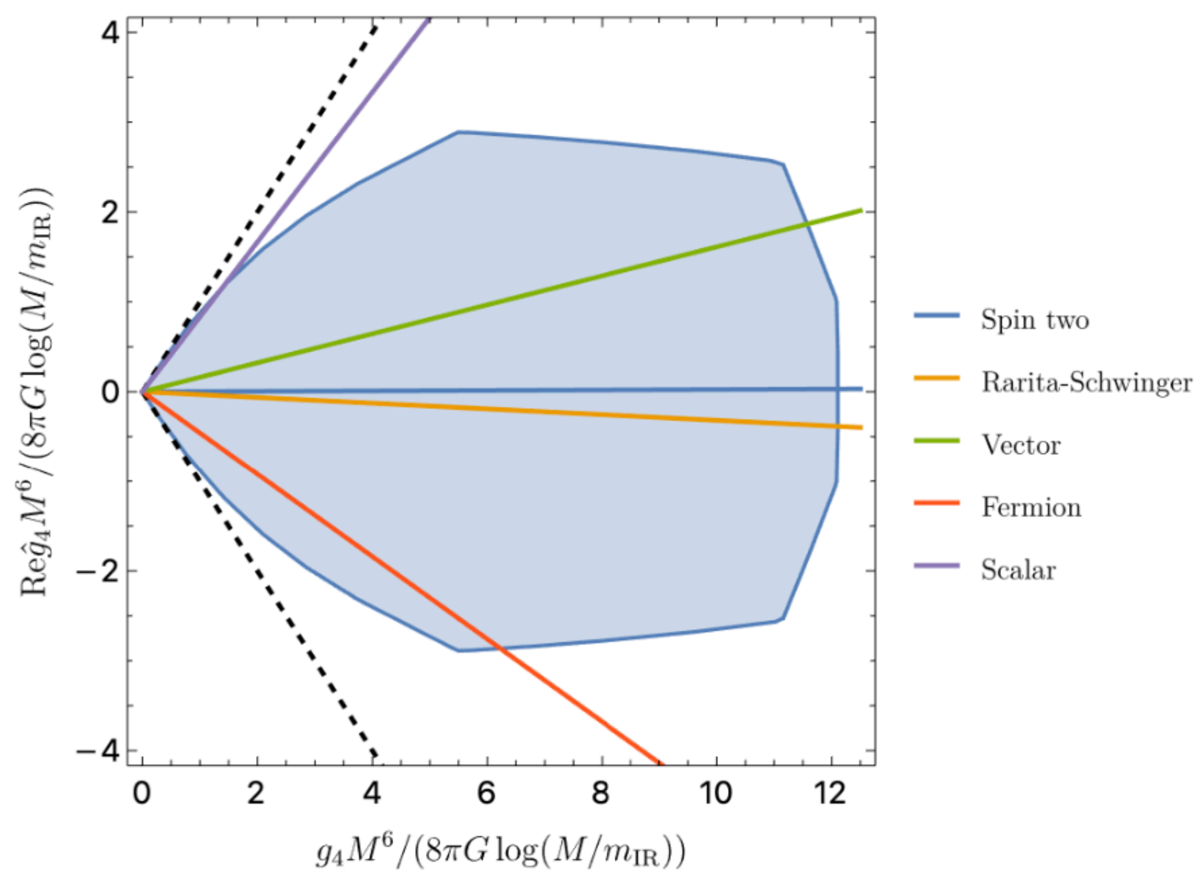
Caron-Huot, Mazac, Rastelli, Simmons-Duffin,2102.08951

Sharp bounds on gravitational EFT

EFT of Einstein gravity

Caron-hot, Li, Parra-Martinez & Simmons-Duffin, 2201.06602
Chiang, Huang, Li, Rodina & Weng, 2201.07177





EFT of Einstein-Maxwell

Henriksson, McPeak, Russo & Vichi, 2203.08164

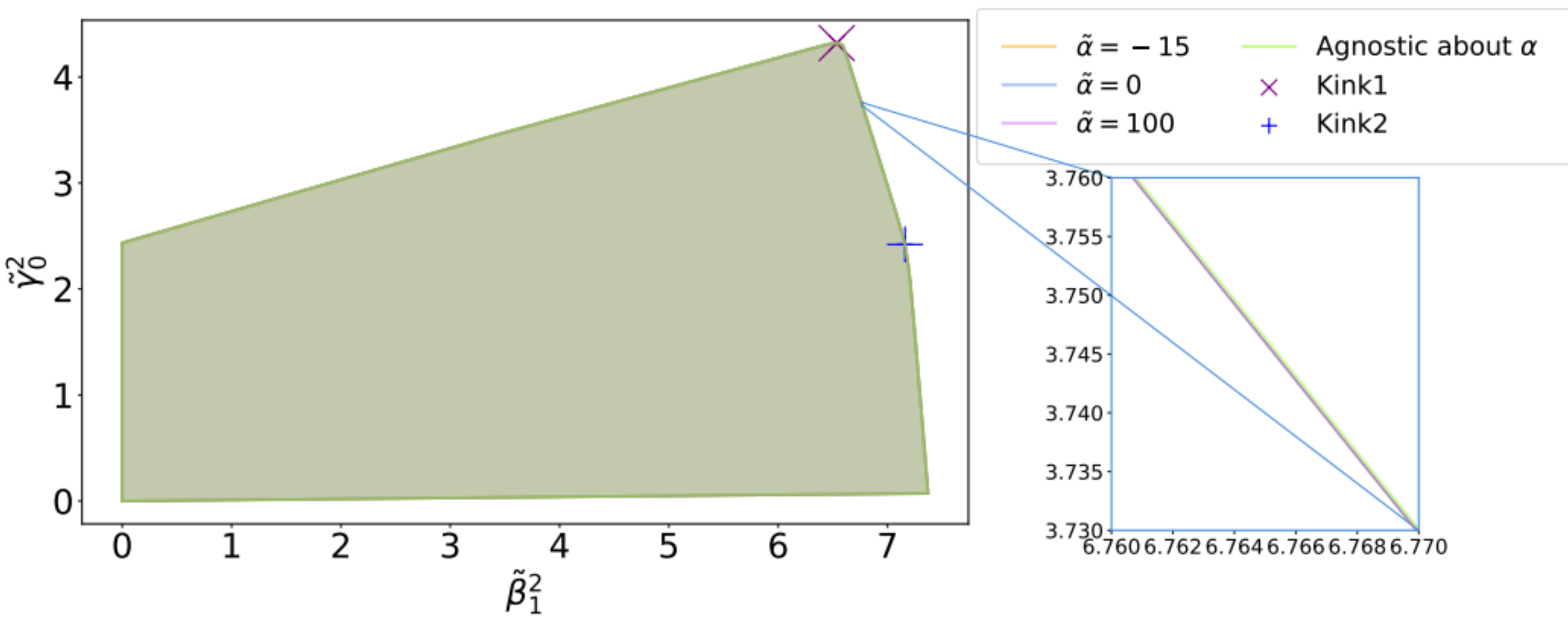
cannot prove weak gravity conjecture with positivity bounds

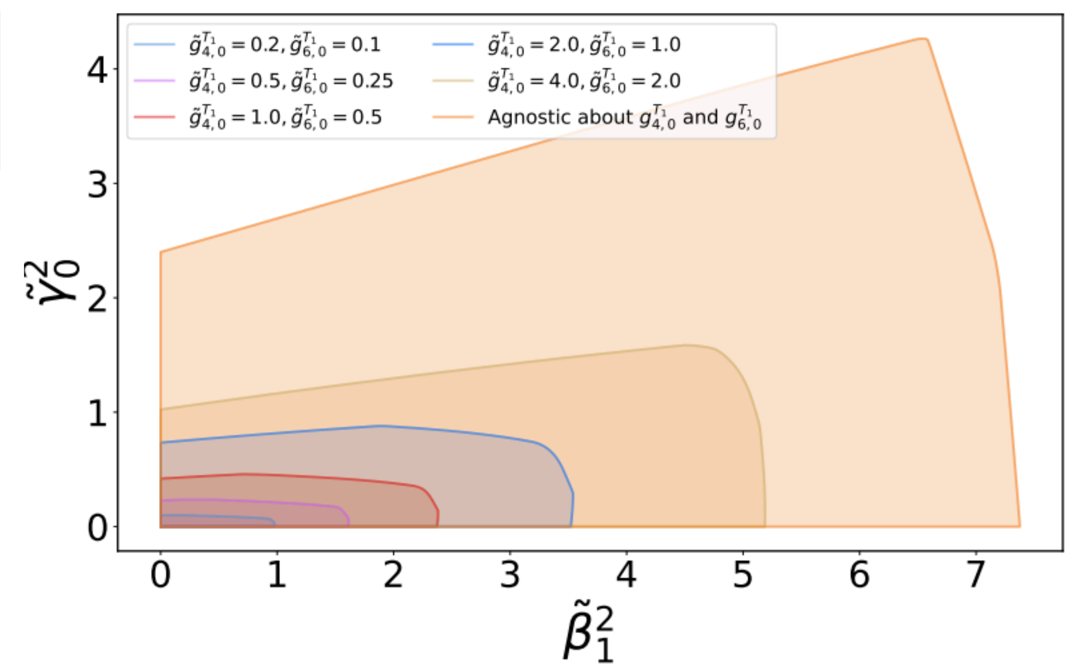
Bounds on scalar-tensor theory

Hong, Wang, **SYZ**, 2304.01259

Tests of GR in strong gravity with gravitational waves

$$S=\int \mathrm{d}^4x\sqrt{-g}\Big(\frac{M_P^2}{2}R-\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi-\frac{\lambda_3}{3!}\phi^3-\frac{\lambda_4}{4!}\phi^4+\frac{\alpha}{2}(\nabla_\mu\phi\nabla^\mu\phi)^2+\frac{\beta_1}{2!}\phi\mathcal{G}+\frac{\beta_2}{4}\phi^2\mathcal{G}$$
$$+\frac{\gamma_0}{3!}\mathcal{R}^{(3)}+\frac{\gamma_1}{3!}\phi\mathcal{R}^{(3)}+\frac{\gamma_2}{2}\nabla_\mu\phi\nabla^\mu\phi\mathcal{R}^{(2)}-\frac{4\gamma_3}{3}\nabla_\mu\phi\nabla_\rho\phi\nabla_\nu\nabla_\sigma\phi R^{\mu\nu\rho\sigma}+\dots\Big)$$





$\phi\mathcal{G}$ and $\phi^2\mathcal{G}$ generate hair BHs and spontaneous scalarization

$$\mathcal{L}\supset M_P^2\sqrt{-g}\Big(\frac{\mathcal{O}(1)}{\Lambda^2}\varphi\mathcal{G}+\frac{\mathcal{O}(1)M_P}{\Lambda^3}\varphi^2\mathcal{G}\Big)$$

scalarization is natural!

Positivity bounds in SMEFT

$$A_{ijkl}(s,t) \sim c^{2,0}_{ijkl} s^2 + c^{2,1}_{ijkl} s^2 t + c^{2,2}_{ijkl} s^2 t^2 + \cdots$$

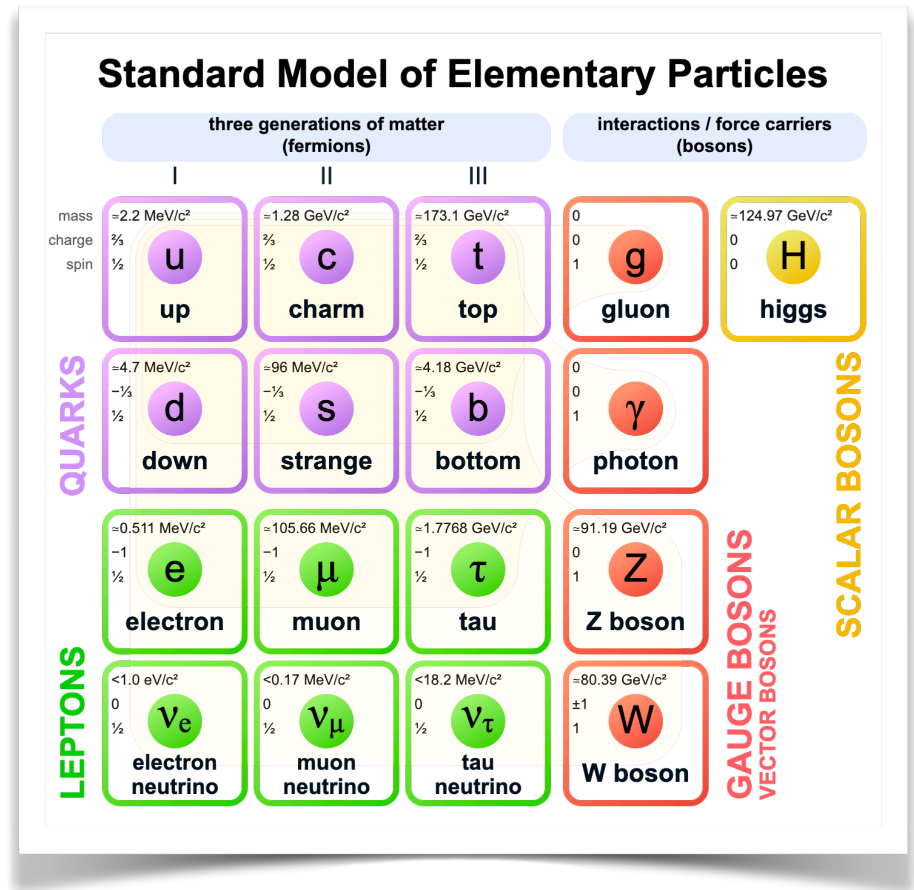
- lowest order positivity bounds: dim-8 or (dim-6)^2
- phenomenologically more relevant

SMEFT: large DoFs

SM Effective Field Theory (SMEFT)

$$\mathscr{L}_{\mathrm{SMEFT}} = \mathscr{L}_{\mathrm{SM}} + \sum_j \frac{c_j^{(6)} O_j^{(6)}}{\Lambda^2} + \sum_i \frac{c_i^{(8)} O_i^{(8)}}{\Lambda^4} + \cdots$$

- SM particle contents
- SM symmetries



- Parametrize new physics
- Popular current approach

if consider up to dim-8, or order s^2 **still huge parameter space!**

How much can positivity bounds reduce the parameter space?

Elastic (forward) positivity bounds

$$A_{ijkl}(s,t) \sim c^{2,0}_{ijkl} s^2 + c^{2,1}_{ijkl} s^2 t + c^{2,2}_{ijkl} s^2 t^2 + \cdots$$

Elastic scattering: particle i + particle j → particle i + particle j

$$M^{ijij} = c^{ijij}_{2,0} > 0$$

use optical theorem

Generalized elastic scattering: $a + b \rightarrow a + b$

superposed states $|a\rangle = \sum_i u_i |i\rangle, \quad |b\rangle = \sum_j v_j |j\rangle$

massless limit

u_i, v_i: arbitrary constants

$$M^{abab} = \sum_{ijkl} u_i v_j u_k^* v_l^* M^{ijkl} = \sum_{ijkl} u_i v_j u_k^* v_l^* c^{ijkl}_{2,0} > 0$$

First example: Vector boson scattering

$$V_1 + V_2 \to V_3 + V_4, \quad V_i \in \{Z, W^+, W^-, \gamma\}$$

$$O_{S,0} = [(D_\mu\Phi)^\dagger D_\nu\Phi] \times [(D^\mu\Phi)^\dagger D^\nu\Phi]$$
$$O_{S,1} = [(D_\mu\Phi)^\dagger D^\mu\Phi] \times [(D_\nu\Phi)^\dagger D^\nu\Phi]$$
$$O_{S,2} = [(D_\mu\Phi)^\dagger D_\nu\Phi] \times [(D^\nu\Phi)^\dagger D^\mu\Phi]$$
$$O_{M,0} = \mathrm{Tr}\left[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}\right] \times \left[(D_\beta\Phi)^\dagger D^\beta\Phi\right]$$
$$O_{M,1} = \mathrm{Tr}\left[\hat{W}_{\mu\nu}\hat{W}^{\nu\beta}\right] \times \left[(D_\beta\Phi)^\dagger D^\mu\Phi\right]$$
$$O_{M,2} = \left[\hat{B}_{\mu\nu}\hat{B}^{\mu\nu}\right] \times \left[(D_\beta\Phi)^\dagger D^\beta\Phi\right]$$
$$O_{M,3} = \left[\hat{B}_{\mu\nu}\hat{B}^{\nu\beta}\right] \times \left[(D_\beta\Phi)^\dagger D^\mu\Phi\right]$$
$$O_{M,4} = \left[(D_\mu\Phi)^\dagger\hat{W}_{\beta\nu}D^\mu\Phi\right] \times \hat{B}^{\beta\nu}$$
$$O_{M,5} = \left[(D_\mu\Phi)^\dagger\hat{W}_{\beta\nu}D^\nu\Phi\right] \times \hat{B}^{\beta\mu}$$
$$O_{M,7} = \left[(D_\mu\Phi)^\dagger\hat{W}_{\beta\nu}\hat{W}^{\beta\mu}D^\nu\Phi\right]$$

$$O_{T,0} = \mathrm{Tr}\left[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}\right] \times \mathrm{Tr}\left[\hat{W}_{\alpha\beta}\hat{W}^{\alpha\beta}\right]$$
$$O_{T,1} = \mathrm{Tr}\left[\hat{W}_{\alpha\nu}\hat{W}^{\mu\beta}\right] \times \mathrm{Tr}\left[\hat{W}_{\mu\beta}\hat{W}^{\alpha\nu}\right]$$
$$O_{T,2} = \mathrm{Tr}\left[\hat{W}_{\alpha\mu}\hat{W}^{\mu\beta}\right] \times \mathrm{Tr}\left[\hat{W}_{\beta\nu}\hat{W}^{\nu\alpha}\right]$$
$$O_{T,5} = \mathrm{Tr}\left[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}\right] \times \hat{B}_{\alpha\beta}\hat{B}^{\alpha\beta}$$
$$O_{T,6} = \mathrm{Tr}\left[\hat{W}_{\alpha\nu}\hat{W}^{\mu\beta}\right] \times \hat{B}_{\mu\beta}\hat{B}^{\alpha\nu}$$
$$O_{T,7} = \mathrm{Tr}\left[\hat{W}_{\alpha\mu}\hat{W}^{\mu\beta}\right] \times \hat{B}_{\beta\nu}\hat{B}^{\nu\alpha}$$
$$O_{T,8} = \hat{B}_{\mu\nu}\hat{B}^{\mu\nu} \times \hat{B}_{\alpha\beta}\hat{B}^{\alpha\beta}$$
$$O_{T,9} = \hat{B}_{\alpha\mu}\hat{B}^{\mu\beta} \times \hat{B}_{\beta\nu}\hat{B}^{\nu\alpha},$$
$$O_{T,10} = \mathrm{Tr}[\hat{W}_{\mu\nu}\tilde{W}^{\mu\nu}]\mathrm{Tr}[\hat{W}_{\alpha\beta}\tilde{W}^{\alpha\beta}],$$
$$O_{T,11} = \mathrm{Tr}[\hat{W}_{\mu\nu}\tilde{W}^{\mu\nu}]\hat{B}_{\alpha\beta}\tilde{B}^{\alpha\beta}.$$

They lead to anomalous Quartic Gauge Couplings (aQGCs)

dim-6 ops require dim-8 ops

If dim-6 ops alone, all elastic positivity bounds are violated

$$\mathcal{O}(\Lambda^{-4}) : \qquad \sum_i (-C_i)\left(\sum_j D_j f_j^{(6)}\right)^2 \geq 0, \quad C_i > 0$$

Positivity bounds require the existence of higher dim ops!

(dim-8 part) - (dim-6 part) > 0

$\mathcal{O}(\Lambda^{-4})$:

weak but simpler bounds

(dim-8 part) > 0 →

$$\sum_i E_i f_i^{(8)} \geq 0$$

Zhang & **SYZ,**1808.00010

Elastic positivity bounds on aQGCs

$$M_{S,ij}F_{S,j} > 0$$
$$M_{M,ij}F_{M,j} > 0$$
$$M_{T,ij}F_{T,j} > 0$$

$$M^{abab} = \Sigma_{ijkl} u_i v_j u_k^* v_l^* M^{ijkl} > 0$$

$$F_{S,i} = (F_{S,0}, F_{S,1}, F_{S,2})^T$$
$$F_{M,i} = (F_{M,0}, F_{M,1}, F_{M,2}, F_{M,3}, F_{M,4}, F_{M,5}, F_{M,7})^T$$
$$F_{T,i} = (F_{T,0}, F_{T,1}, F_{T,2}, F_{T,5}, F_{T,6}, F_{T,7}, F_{T,8}, F_{T,9})^T$$

$$M_S = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$M_M = \begin{pmatrix} 0 & -2c_W^4 & 0 & -s_W^4 & 0 & s_W^2 c_W^2 & c_W^4 \\ 0 & -2c_W^4 & 0 & -s_W^4 & 0 & -s_W^2 c_W^2 & c_W^4 \\ 0 & -2 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & -1 & 0 & 1 & 1 \\ 0 & -2 & 0 & -1 & 0 & -1 & 1 \end{pmatrix}$$

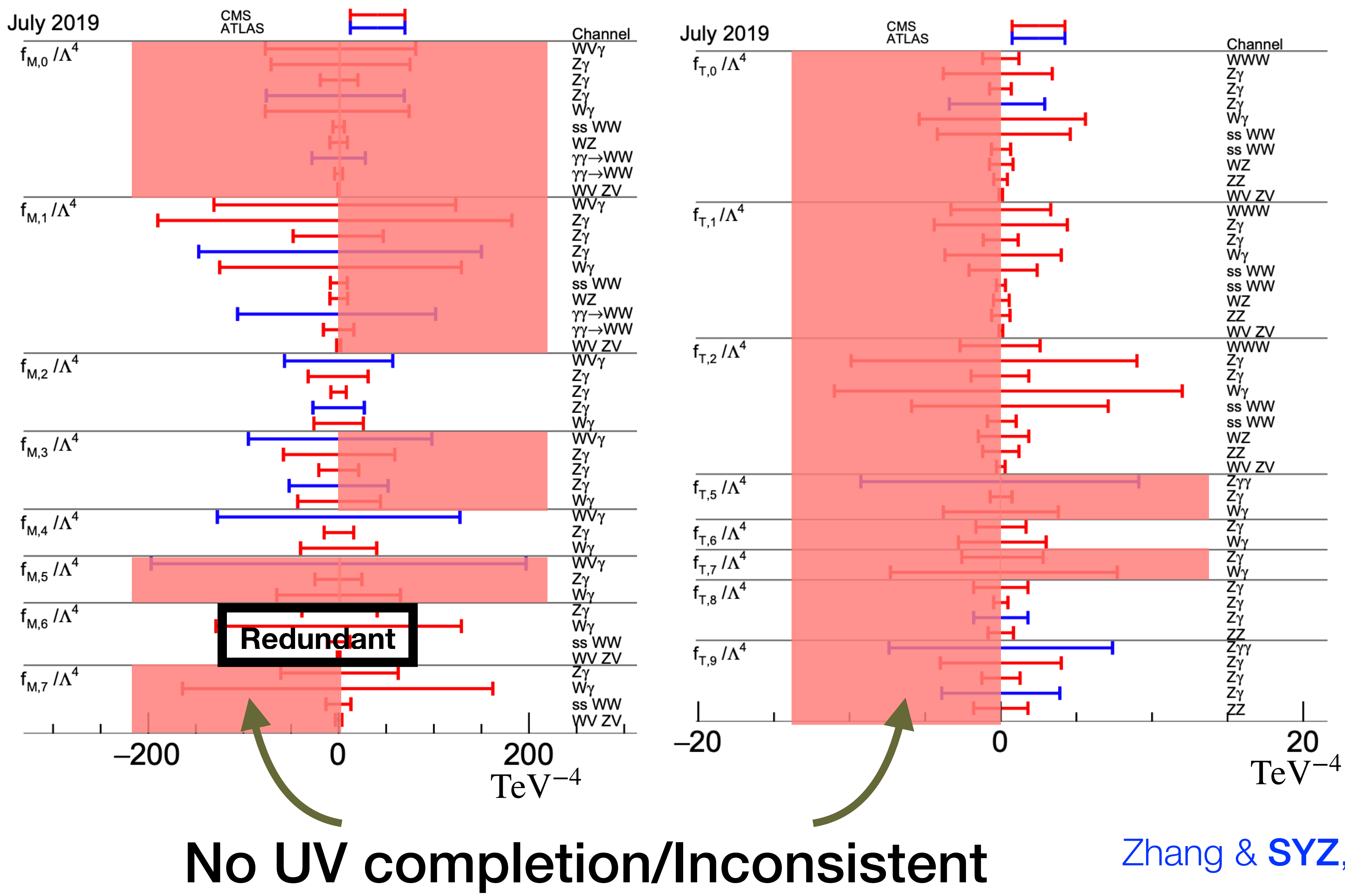
$$M_T = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 8 & 12 & 5 & 0 & 0 & 0 & 0 & 0 \\ 8c_W^8 & 8c_W^8 & 4c_W^8 & 4c_W^4 s_W^4 & 4c_W^4 s_W^4 & 2c_W^4 s_W^4 & 2s_W^8 & s_W^8 \\ 0 & 0 & 4c_W^8 & 0 & 0 & 2c_W^4 s_W^4 & 0 & s_W^8 \\ 0 & 0 & 4c_W^4 & 0 & 0 & s_W^4 & 0 & 0 \\ 0 & 16c_W^4 & 4c_W^4 & 0 & 4s_W^4 & s_W^4 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 16 & 4 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 8c_W^4 & 0 & 0 & \left(c_W^2 - s_W^2\right)^2 & 0 & 2s_W^4 \\ 32c_W^4 & 32c_W^4 & 16c_W^4 & -16c_W^2 s_W^2 & 4\left(c_W^2 - s_W^2\right)^2 & 1 - 8s_W^2 c_W^2 & 8s_W^4 & 4s_W^4 \\ 0 & 0 & 4 & 0 & 0 & 2 & 0 & 1 \\ 8 & 8 & 4 & 4 & 4 & 2 & 2 & 1 \end{pmatrix}$$

+ a few nonlinear bounds

Bi, Zhang & **SYZ**,1902.08977

Positivity vs experimental bounds (1D bounds)

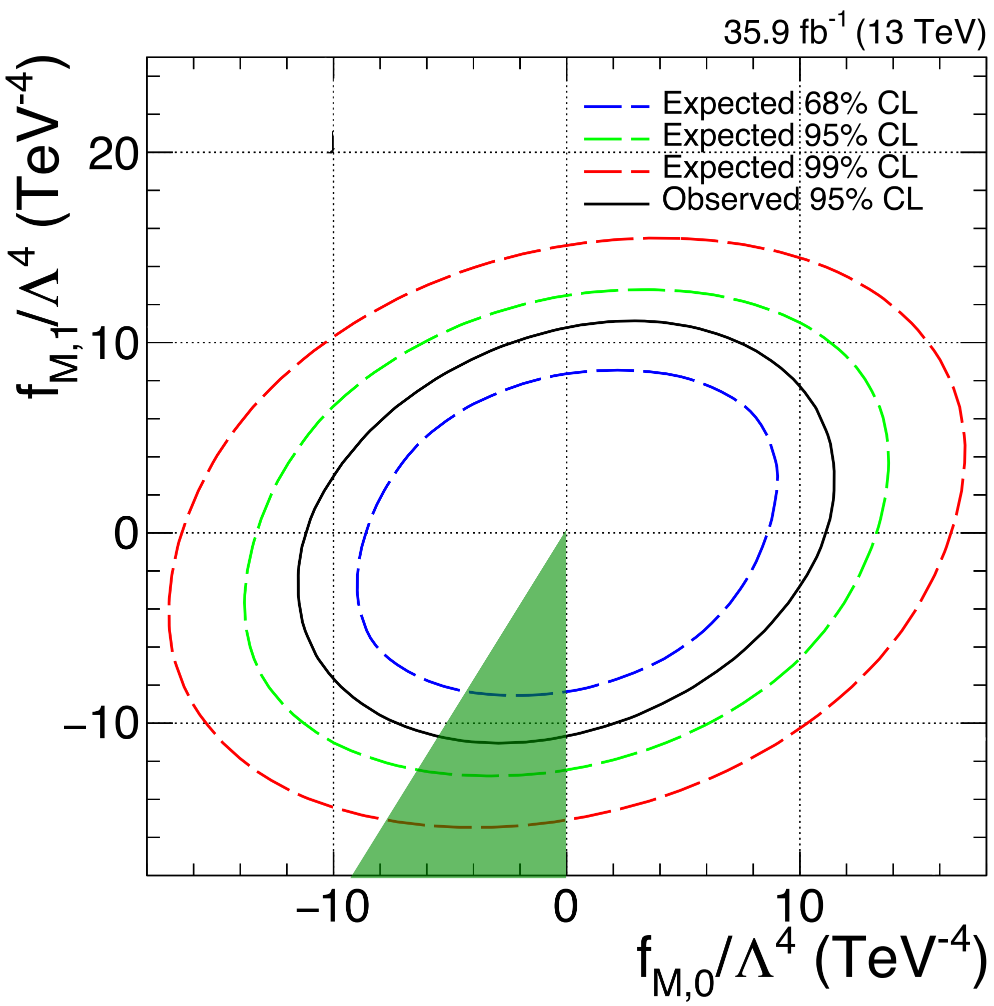
Constraining aGQC in VBS



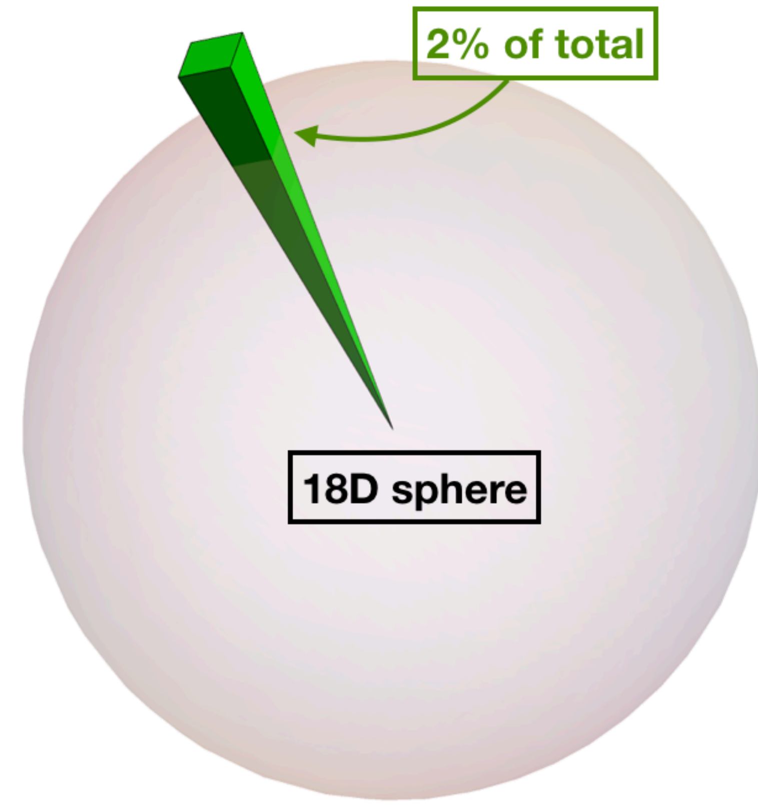
Zhang & **SYZ**,1808.00010

Higher dimensional bounds

O_{M0} and O_{M1}



Space of 18 Wilson coeffs for aQGCs



Only **~2%** of the total aQGC parameter space admits an analytic UV completion!

Zhang & **SYZ**,1808.00010

Example: Flavor constraints from positivity

Consider 4-fermion scattering

Flavor-violating couplings bounded by flavor-conserving couplings

eg: $|c_{1231}|^2 < c_{1221}c_{1331}$

lepton no, strong isospin, strangeness, CP, etc

$$\mathcal{O}_1[\psi] = -c^{\psi,1}_{mnpq}\partial_\mu J_\nu[\psi]_{mn}\partial^\mu J^\nu[\psi]_{pq},$$
$$\mathcal{O}_2[\psi] = -c^{\psi,2}_{mnpq}\partial_\mu J_\nu[\psi]^I_{mn}\partial^\mu J^\nu[\psi]^I_{pq},$$
$$\mathcal{O}_3[\psi] = -c^{\psi,3}_{mnpq}\partial_\mu J_\nu[\psi]^a_{mn}\partial^\mu J^\nu[\psi]^a_{pq},$$
$$\mathcal{O}_4[Q] = -c^{Q,4}_{mnpq}\partial_\mu J_\nu[Q]^{Ia}_{mn}\partial^\mu J^\nu[Q]^{Ia}_{pq}.$$
$$\mathcal{O}_{J1}[\psi,\chi] = -b^{\psi\chi,1}_{mnpq}\partial_\mu J_\nu[\psi]_{mq}\partial^\mu J^\nu[\chi]_{np},$$
$$\mathcal{O}_{J2}[Q,L] = -b^{QL,2}_{mnpq}\partial_\mu J_\nu[Q]^I_{mq}\partial^\mu J^\nu[L]^I_{np},$$
$$\mathcal{O}_{J3}[\psi,\chi] = -b^{\psi\chi,3}_{mnpq}\partial_\mu J_\nu[\psi]^a_{mq}\partial^\mu J^\nu[\chi]^a_{np},$$
$$\mathcal{O}_{K1}[\psi,\chi] = -a^{\psi\chi,1}_{mnpq}K_{\mu\nu}[\psi]_{mq}K^{\nu\mu}[\chi]_{np},$$
$$\mathcal{O}_{K2}[Q,L] = -a^{QL,2}_{mnpq}K_{\mu\nu}[Q]^I_{mq}K^{\nu\mu}[L]^I_{np}$$
$$\mathcal{O}_{K3}[\psi,\chi] = -a^{\psi\chi,3}_{mnpq}K_{\mu\nu}[\psi]^a_{mq}K^{\nu\mu}[\chi]^a_{np},$$

Application   Rodd & Remmen, PRL, 2004.02885

eg: Mu3e experiment targets $c^{e,1}_{1112}$ with

$\mathrm{Br}(\mu \to 3e) \sim 10^{-16} - 10^{-12}$ requires $c^{e,1}_{1112} \gtrsim (300\,\mathrm{GeV})^{-4}$

Positivity: $|c^{e,1}_{1112}|^2 < c^{e,1}_{1111}c^{e,1}_{2112}$ LEP: $c^{e,1}_{2112}, c^{e,1}_{1111} \lesssim (500\,\mathrm{GeV})^{-4}$

LEP preclude certain operators in upcoming $\mu \to 3e$ experiment

Stronger positivity bounds?

$$M^{abab} = \sum_{ijkl} u_i v_j u_k^* v_l^* M^{ijkl} = \sum_{ijkl} u_i v_j u_k^* v_l^* c_{2,0}^{ijkl} > 0$$

Is it possible such that

$$M^T = \sum_{ijkl} T_{ijkl} M^{ijkl} > 0, \text{ and } \{T_{ijkl}\} \supset \{u_i v_j u_k^* v_l^*\}?$$

Yes, T_{ijkl} is more than $u_i v_j u_k^* v_l^*$!

Example: W-boson scatterings in SMEFT

$$F_{T,2} \geq 0, \quad 4F_{T,1} + F_{T,2} \geq 0$$
$$F_{T,2} + 8F_{T,10} \geq 0, \quad 8F_{T,0} + 4F_{T,1} + 3F_{T,2} \geq 0$$
$$12F_{T,0} + 4F_{T,1} + 5F_{T,2} + 4F_{T,10} \geq 0$$
$$4F_{T,0} + 4F_{T,1} + 3F_{T,2} + 12F_{T,10} \geq 0$$

Zhang & **SYZ**, PRL, 2005.03047

old: $|a\rangle|b\rangle \rightarrow |a\rangle|b\rangle$

new: $|U\rangle \rightarrow |U\rangle$

scatterings of entangled states

$$T_{ijkl} \sim \Sigma_n \lambda_n U_{ij}^n U_{kl}^n$$

Best bounds from ERs of $\mathscr{T}$ cone

generalized optical theorem

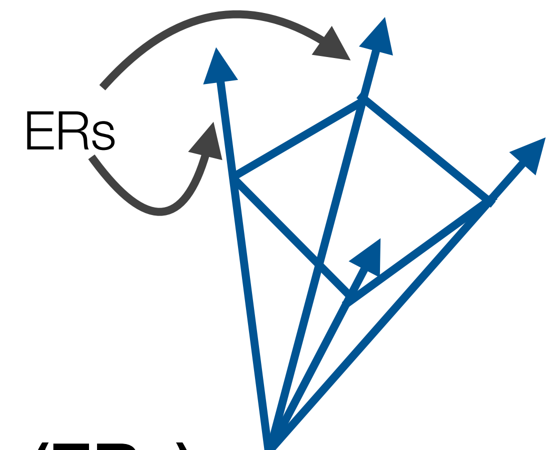
$$T_{ijkl} \in \mathscr{T} \equiv \mathscr{T}^+ \cap \overrightarrow{\mathbf{S}} \quad \begin{cases} \mathscr{T}^+ \equiv \left\{ T_{ijkl} \mid T_{ij,kl} \succeq 0 \right\} \\ \overrightarrow{\mathbf{S}} \equiv \left\{ T_{ijkl} \mid T_{ijkl} = T_{ilkj} = T_{kjil} = T_{jilk} \right\} \end{cases}$$

$\mathscr{T}$ is a spectrahedron

Li, Xu, Yang, Zhang & **SYZ**, PRL, 2101.01191

(spectrahedron) = (convex cone of PSD matrices) $\cap$ affine-linear space

To get best bounds, find all ERs of $\mathscr{T}$



all elements of $\mathscr{T}$: $T_{ijkl} = \Sigma_p \alpha_p T^{(p)}_{ijkl}, \;\; \alpha_p > 0$

p enumerates all **Extreme Rays (ERs)**

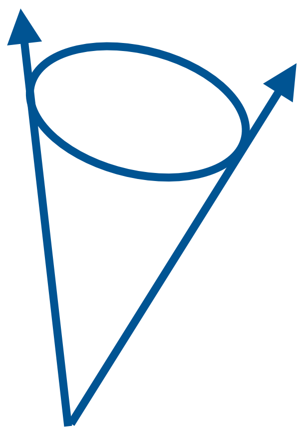
Best positivity bounds:

$$\sum_{ijkl} T^{(p)}_{ijkl} M^{ijkl} > 0$$

Semi-definite program (SDP)

spectrahedron is parameter space of a semi-definite program

Use SDP to find best positivity bounds



minimize: $\sum_{ijkl} T_{ijkl} M^{ijkl}$

subject to: $T_{ijkl} \in \mathscr{T} \equiv \mathscr{T}^+ \cap \overrightarrow{\mathbf{S}}$

$\min(T \cdot M) > 0$, then M^{ijkl} is within positivity bounds

Compared to elastic approach ($uvuvM > 0$)

- **stronger bounds**
- **more efficient (polynomial complexity)**

Li, Xu, Yang, Zhang & **SYZ**, PRL, 2101.01191

Convex cone $\mathscr{C}$ of amplitudes

$$\mathscr{C} \equiv \{M^{ijkl}\} = \mathrm{cone}\left(\left\{m^{i(j}m^{|k|l)}\right\}\right)$$

$m^{ij} \sim M^{ij\to X}$

X: intermediate state

$\mathscr{C}$ is dual cone of $\mathscr{T}$: $\mathscr{T} \equiv \left\{T^{ijkl} \,\middle|\, T\cdot M \equiv \sum_{ijkl} T_{ijkl}M^{ijkl} > 0\right\}$

For m^{ij} to be extremal, it can not be split to two amplitudes

$$m^{ij}_{(\mathrm{ER})} \sim M^{ij\to X_{\mathrm{irrep}}} \sim C^{r,\alpha}_{i,j}$$

CG coefficient

Get $\mathscr{C}$ cone by symmetries of EFT

$$\mathscr{C} = \mathrm{cone}\left(\left\{P_r^{i(j|k|l)}\right\}\right)$$

$$P_r^{ijkl} \equiv \sum_\alpha C^{r,\alpha}_{i,j}\left(C^{r,\alpha}_{k,l}\right)^*$$

group projector

Zhang & **SYZ**, PRL, 2005.03047

The inverse problem

Structure of $\mathscr{C}$ cone implies

Zhang & **SYZ**, PRL, 2005.03047

Extremal Ray ⟷ **UV Particle**

$P_r^{i(j|k|l)}$

Example: Higgs $\mathscr{C}$ cone in SMEFT

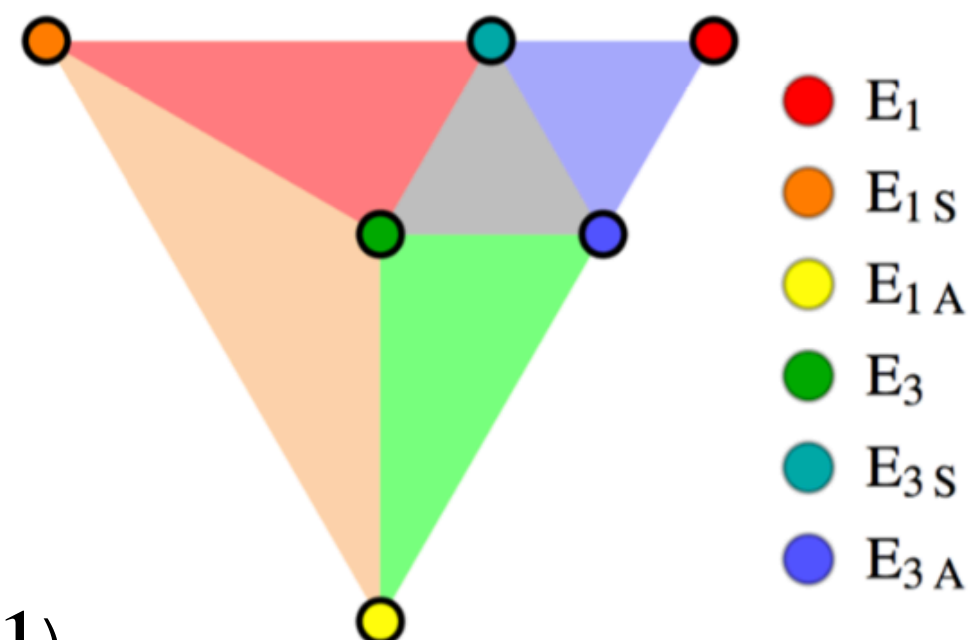
Wilson coeffs fall in blue region

↓

E_1 must exit

↓

new UV state ($SU(2)_L$ singlet, $Y = 1$)



ERs of $\mathscr{C}$ (or dim-8 operators) are important to reverse-engineer UV physics!

Zhang, 2112.11665

VBS and 4-gluon interactions

- Transversal VBS

  10D parameter space: 0.681%

  Yamashita, Zhang & **SYZ**, 2009.04490

$$O_{T,0} = \mathrm{Tr}[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}]\mathrm{Tr}[\hat{W}_{\alpha\beta}\hat{W}^{\alpha\beta}]$$
$$O_{T,2} = \mathrm{Tr}[\hat{W}_{\alpha\mu}\hat{W}^{\mu\beta}]\mathrm{Tr}[\hat{W}_{\beta\nu}\hat{W}^{\nu\alpha}]$$
$$O_{T,5} = \mathrm{Tr}[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}]\hat{B}_{\alpha\beta}\hat{B}^{\alpha\beta}$$
$$O_{T,7} = \mathrm{Tr}[\hat{W}_{\alpha\mu}\hat{W}^{\mu\beta}]\hat{B}_{\beta\nu}\hat{B}^{\nu\alpha}$$
$$O_{T,8} = \hat{B}_{\mu\nu}\hat{B}^{\mu\nu}\hat{B}_{\alpha\beta}\hat{B}^{\alpha\beta}$$

$$O_{T,1} = \mathrm{Tr}[\hat{W}_{\alpha\nu}\hat{W}^{\mu\beta}]\mathrm{Tr}[\hat{W}_{\mu\beta}\hat{W}^{\alpha\nu}]$$
$$O_{T,10} = \mathrm{Tr}[\hat{W}_{\mu\nu}\tilde{W}^{\mu\nu}]\mathrm{Tr}[\hat{W}_{\alpha\beta}\tilde{W}^{\alpha\beta}]$$
$$O_{T,6} = \mathrm{Tr}[\hat{W}_{\alpha\nu}\hat{W}^{\mu\beta}]\hat{B}_{\mu\beta}\hat{B}^{\alpha\nu}$$
$$O_{T,11} = \mathrm{Tr}[\hat{W}_{\mu\nu}\tilde{W}^{\mu\nu}]\hat{B}_{\alpha\beta}\tilde{B}^{\alpha\beta}$$
$$O_{T,9} = \hat{B}_{\alpha\mu}\hat{B}^{\mu\beta}\hat{B}_{\beta\nu}\hat{B}^{\nu\alpha}$$

- 4-gluon SMEFT operators

  7D parameter space: 1.6628%

  obtained bounds both in $\mathscr{C}$ and $\mathscr{T}$ cone

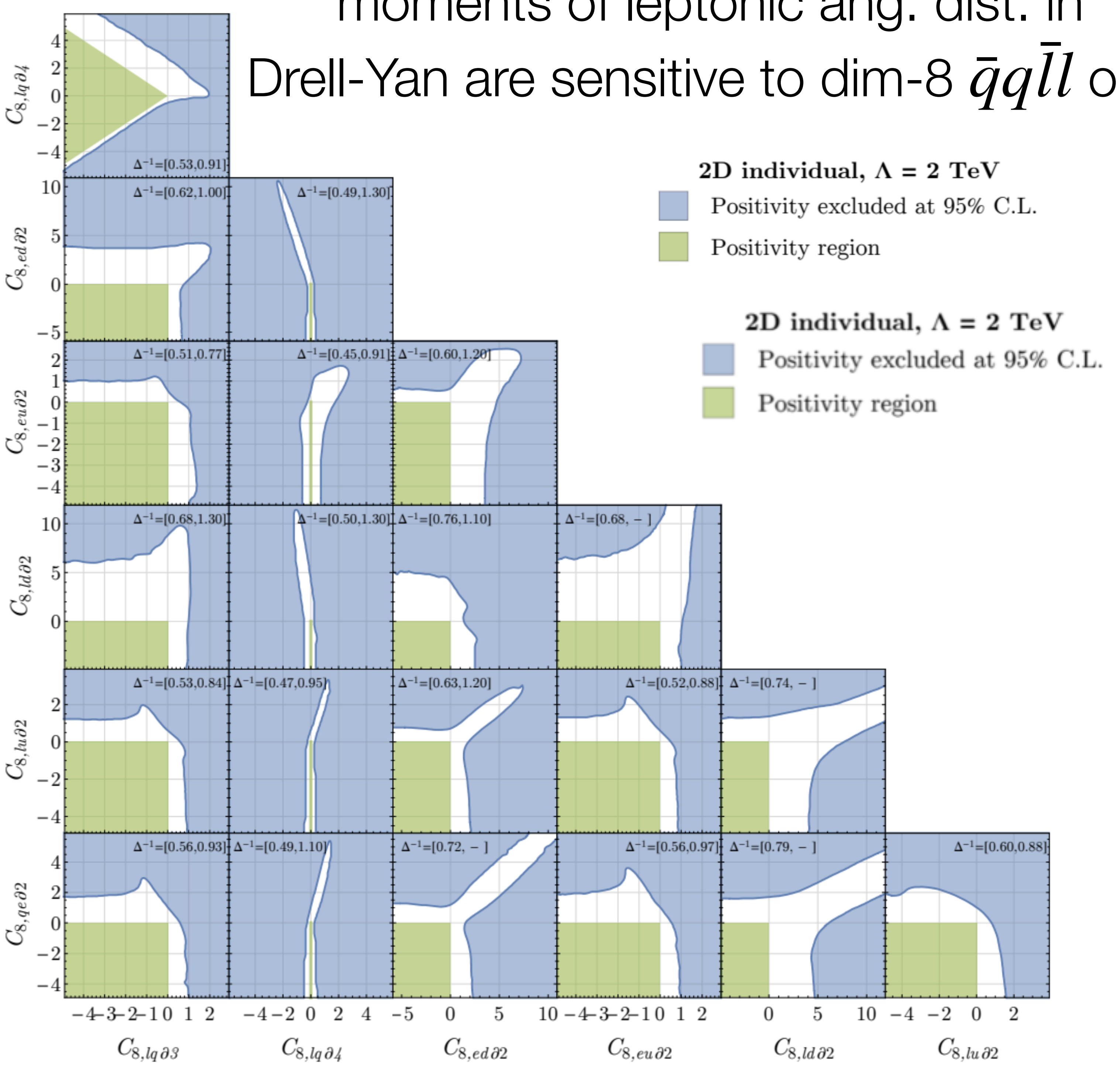
  Li, Xu, Yang, Zhang & **SYZ**, PRL, 2101.01191

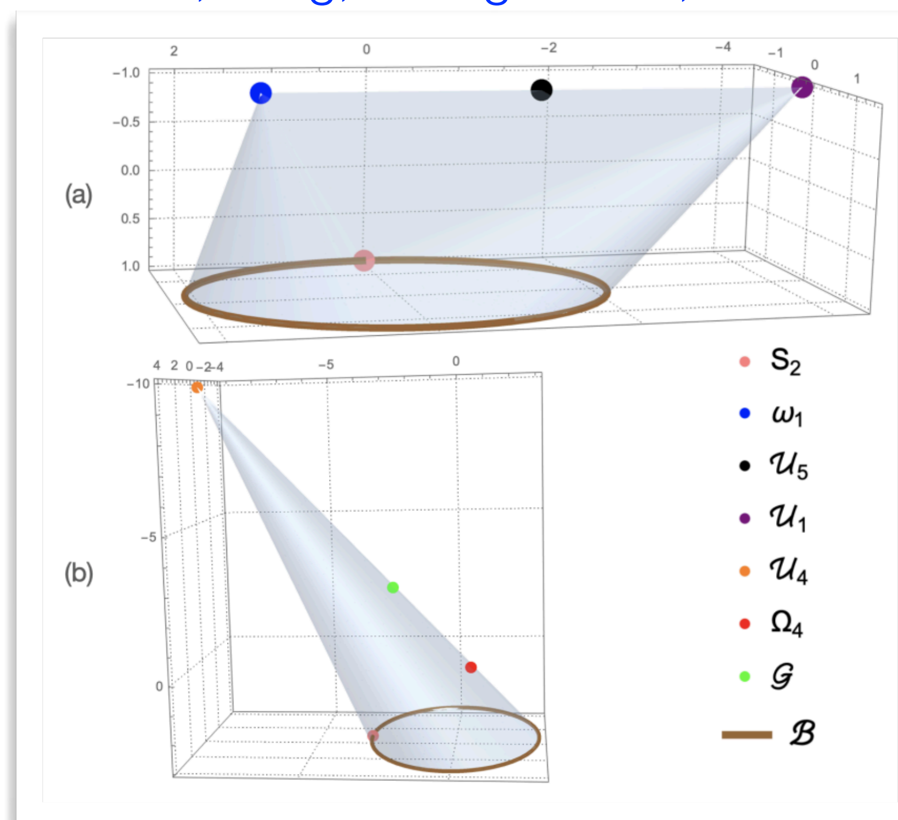
$Q^{(1)}_{G^4}$	$(G^A_{\mu\nu}G^{A\mu\nu})(G^B_{\rho\sigma}G^{B\rho\sigma})$
$Q^{(2)}_{G^4}$	$(G^A_{\mu\nu}\widetilde{G}^{A\mu\nu})(G^B_{\rho\sigma}\widetilde{G}^{B\rho\sigma})$
$Q^{(3)}_{G^4}$	$(G^A_{\mu\nu}G^{B\mu\nu})(G^A_{\rho\sigma}G^{B\rho\sigma})$
$Q^{(4)}_{G^4}$	$(G^A_{\mu\nu}\widetilde{G}^{B\mu\nu})(G^A_{\rho\sigma}\widetilde{G}^{B\rho\sigma})$
$Q^{(7)}_{G^4}$	$d^{ABE}d^{CDE}(G^A_{\mu\nu}G^{B\mu\nu})(G^C_{\rho\sigma}G^{D\rho\sigma})$
$Q^{(8)}_{G^4}$	$d^{ABE}d^{CDE}(G^A_{\mu\nu}\widetilde{G}^{B\mu\nu})(G^C_{\rho\sigma}\widetilde{G}^{D\rho\sigma})$
Q_G	$f^{ABC}G^{A\nu}_{\mu}G^{B\rho}_{\nu}G^{C\mu}_{\rho}$

Test positivity with Drell-Yan at LHC

Li, Mimasu, Yamashita, Yang, Zhang & **SYZ**, 2204.13121

moments of leptonic ang. dist. in Drell-Yan are sensitive to dim-8 $\bar{q}q\bar{l}l$ ops



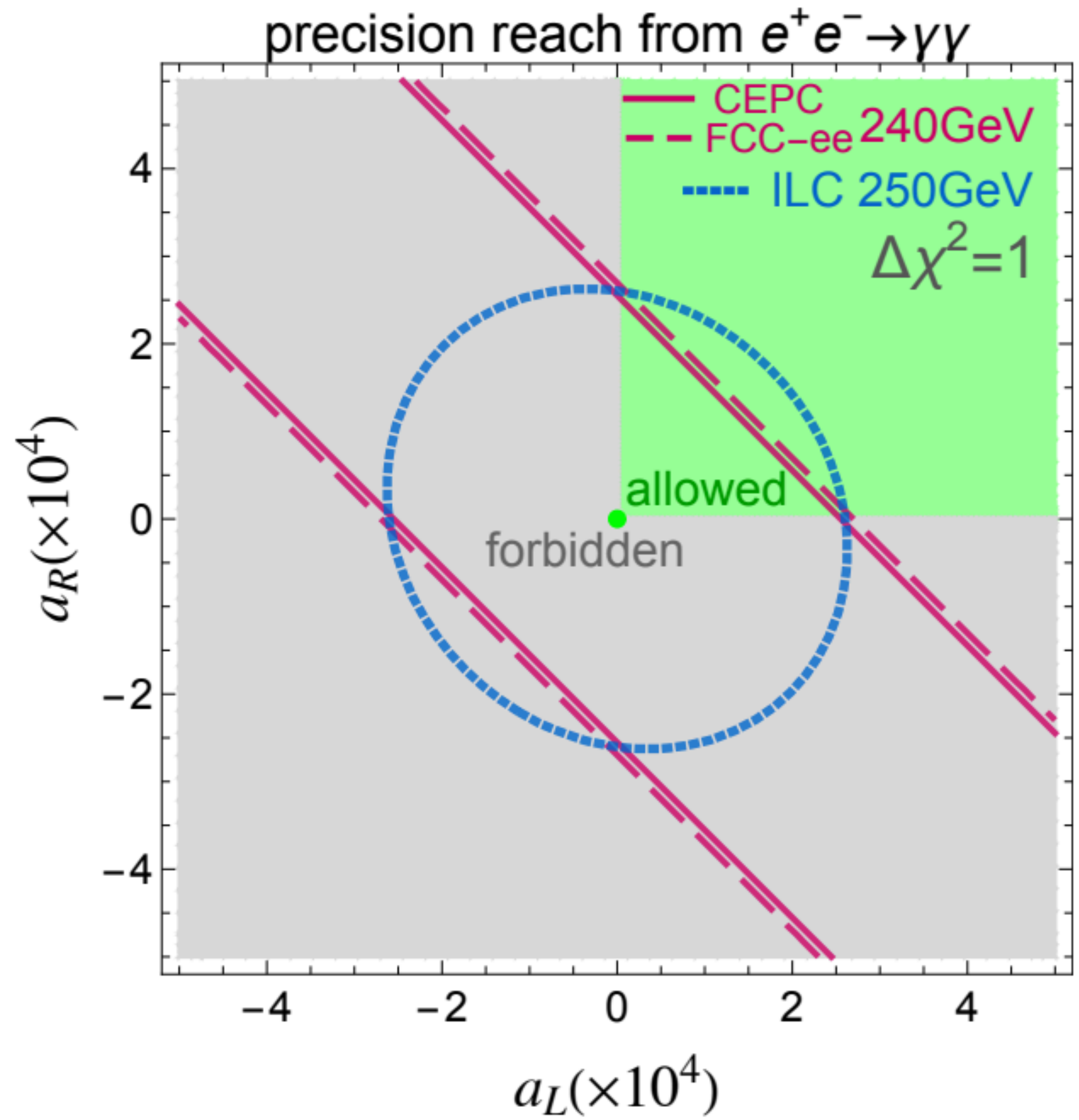


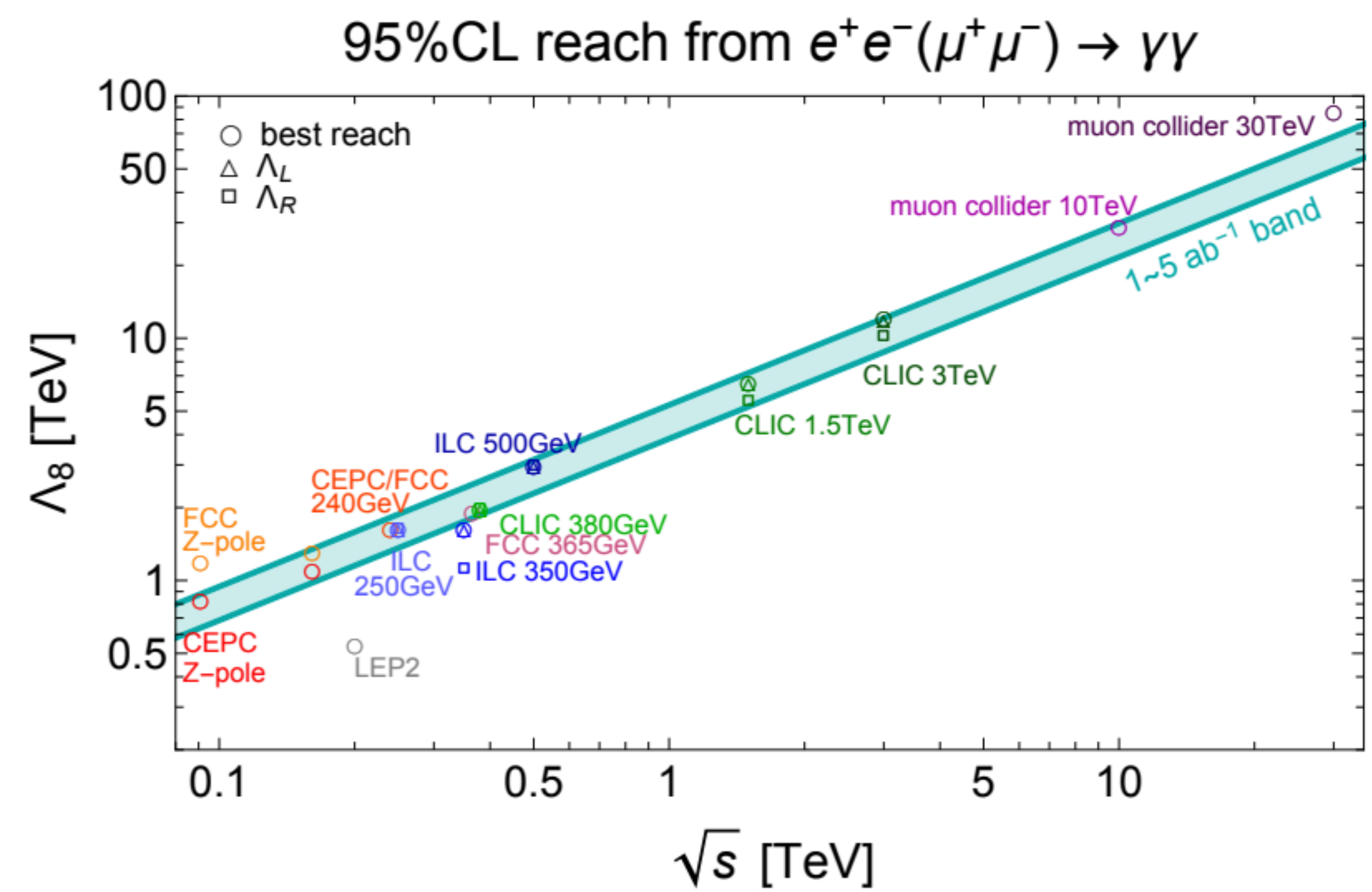
UV particle H	$\lambda_{\max}$ [TeV^{-4}]	$\frac{M_H}{\sqrt{g_H}}$ [TeV]
$\mathcal{S}_2$	0.0015	≥ 5.1
$\mathcal{U}_4$	1.2	≥ 0.95
Ω_4	1.1	≥ 0.97
ω_1	0.092	≥ 1.8
$\mathcal{U}_5$	0.046	≥ 2.2
$\mathcal{B}$	0.00075	≥ 6.1
$\mathcal{G}$	2.5	≥ 0.80
$\mathcal{U}_1$	0.092	≥ 1.8

Unambiguous test of positivity at lepton colliders

Leading diphoton channel is from dim-8 Gu, Wang & Zhang, PRL, 2011.03055

$$e^+e^-(\mu^+\mu^-) \to \gamma\gamma$$





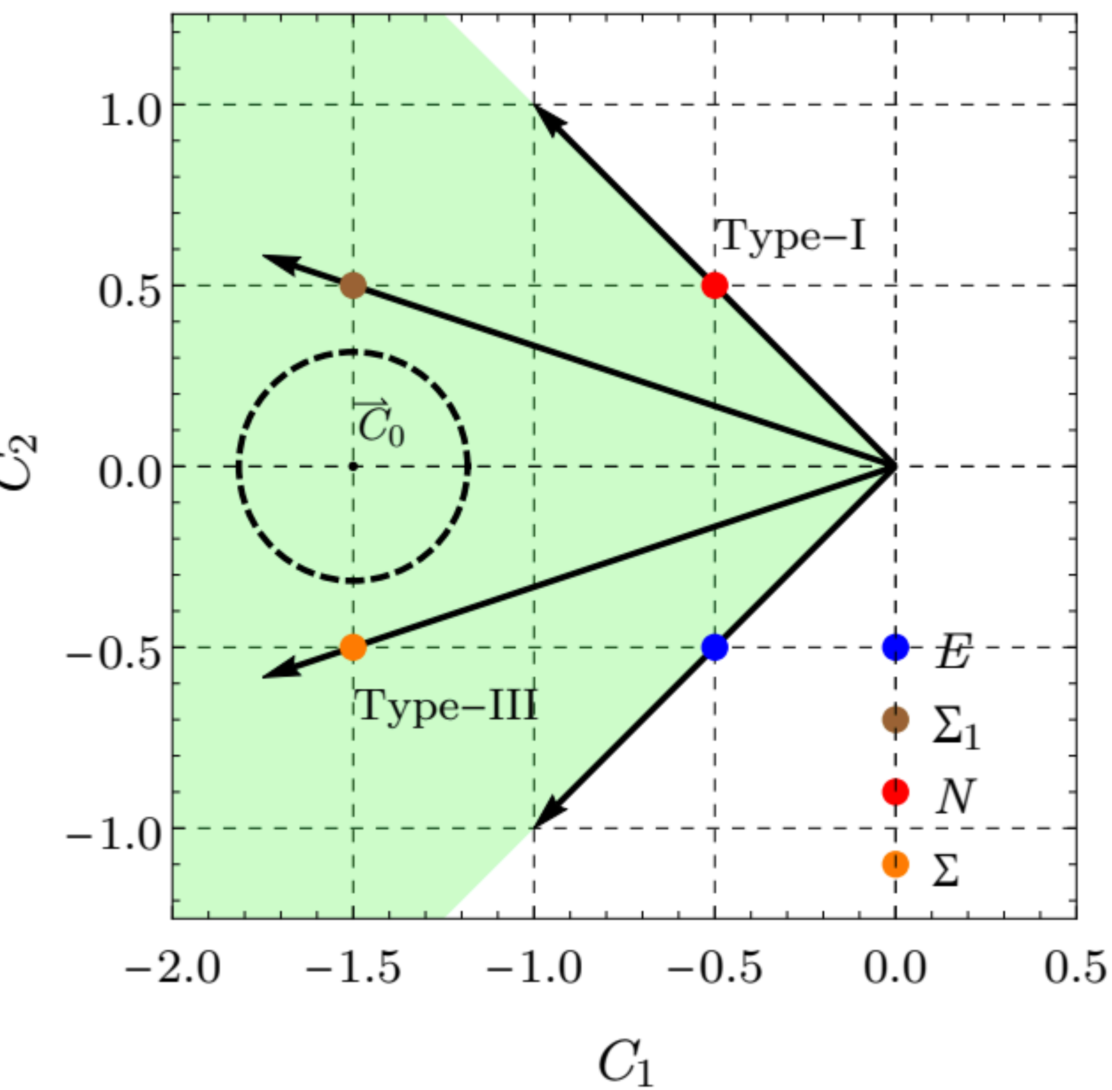
$$a_L = -2\frac{v^4}{\Lambda^4}\left(c_W^2 c_{l^2B^2D} - 2 s_W c_W c^{(2)}_{l^2WBD} + s_W^2 c^{(1)}_{l^2W^2D}\right)$$

$$a_R = -2\frac{v^4}{\Lambda^4}\left(c_W^2 c_{e^2B^2D} + s_W^2 c_{e^2W^2D}\right)$$

Positivity cone and neutrino seesaw types

UV State	Spin	$SU(2)_L \otimes U(1)_Y$	Interaction	Seesaw	Extremal Ray	$\vec{c}$
N	1/2	$\mathbf{1}_0$	$g\bar{N}\left(H^{\mathrm{T}}\epsilon L\right)$	Type-I	✔	$\frac{1}{2}(-1,1,0,0,0,0,0)$
Σ	1/2	$\mathbf{3}_0$	$g\bar{\Sigma}^I\left(H^{\mathrm{T}}\epsilon\sigma^I L\right)$	Type-III	✗	$\frac{1}{2}(-3,-1,0,0,0,0,0)$
Ξ_1	0	$\mathbf{3}_1$	$g\Xi_1^I\left[M(H^\dagger\epsilon\sigma^I H^*)+x(\bar{L}^c\epsilon\sigma^I L)\right]$	Type-II	✗	$\frac{1}{2}(0,0,-3x^2,-x^2,0,16,($

Li & Zhou, 2202.12907



$$\mathcal{O}_1=(\bar{L}\gamma_\mu i\overleftrightarrow{D}_\nu L)\left(D^\mu H^\dagger D^\nu H\right),$$
$$\mathcal{O}_2=(\bar{L}\gamma_\mu\sigma^I i\overleftrightarrow{D}_\nu L)\left(D^\mu H^\dagger\sigma^I D^\nu H\right)$$
$$\mathcal{O}_3=\partial_\nu\left(\bar{L}\gamma^\mu L\right)\partial^\nu\left(\bar{L}\gamma_\mu L\right),$$
$$\mathcal{O}_4=\partial_\nu\left(\bar{L}\gamma^\mu\sigma^I L\right)\partial^\nu\left(\bar{L}\gamma_\mu\sigma^I L\right)$$
$$\mathcal{O}_5=\left(D_\mu H^\dagger D_\nu H\right)\left(D^\nu H^\dagger D^\mu H\right)$$
$$\mathcal{O}_6=\left(D_\mu H^\dagger D_\nu H\right)\left(D^\mu H^\dagger D^\nu H\right)$$
$$\mathcal{O}_7=\left(D_\mu H^\dagger D^\mu H\right)\left(D_\nu H^\dagger D^\nu H\right)$$

only type-I seesaw is an Extremal Ray

type-II and type III live inside positivity cone

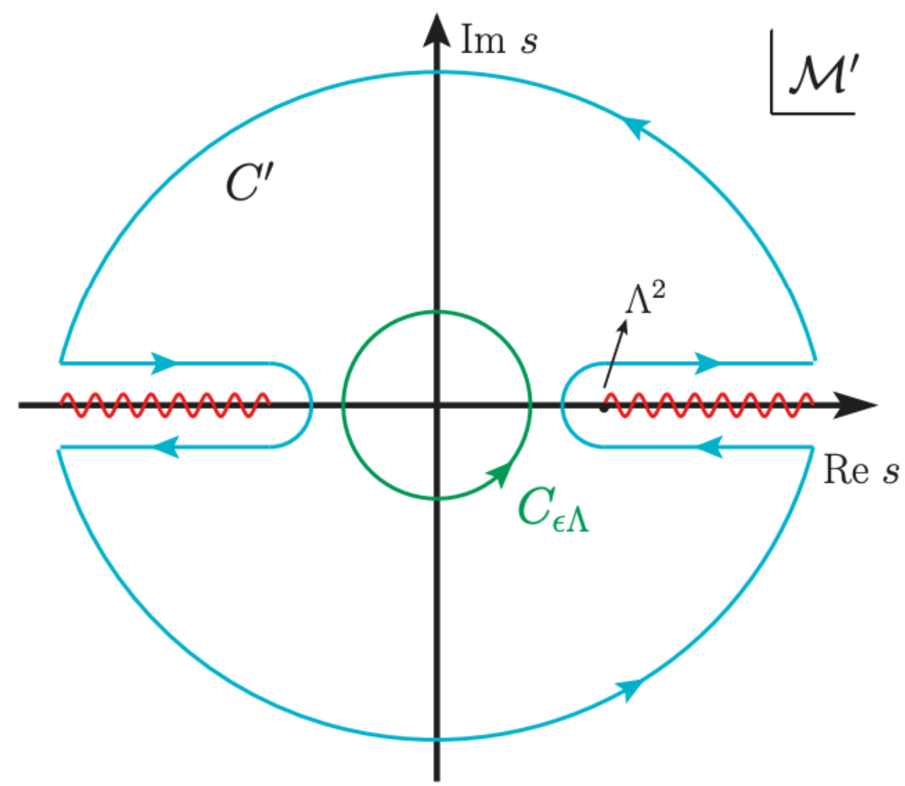
Higgs sector: 1 loop corrections

Li, 2212.12227

Improved positivity bounds

de Rham, Melville, Tolley & SYZ, 1702.08577

$$M'(s) = M(s) - \frac{1}{2\pi i}\int_{-\Lambda^2}^{+\Lambda^2} ds' \frac{\mathrm{Disc}\, M(s')}{s'-s}$$



Positivity cone up to 1-loop

Li, 2212.12227

$$\log(-s) \to \log(-s) - \frac{1}{2\pi i}\int_0^{\Lambda^2} ds' \frac{\mathrm{Disc}\log(-s')}{s'-s} = \log(\Lambda^2 - s)$$

$$C_2' \geq 0, \quad C_1' + C_2' \geq 0, \quad C_1' + C_2' + C_3' \geq 0$$

for example $C_1' = C_{H1}^{(8)} - \frac{\Delta\lambda(36C_{H1}^{(8)} + 13C_{H2}^{(8)} + 13C_{H3}^{(8)} - 18\Delta\lambda)}{72\pi^2} + \frac{13C_{H1}^{(6)2} + 26C_{H1}^{(6)}C_{H2}^{(6)} + 8C_{H2}^{(6)2} - 10C_{H\psi_L}^{(1)2} + 10C_{H\psi_L}^{(3)2} - 5C_{H\psi_R}^2 + 5C_{Hud}^2}{36\pi^2}$

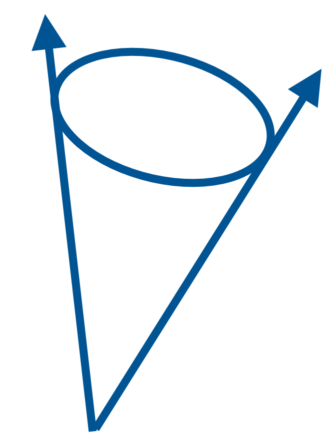
cone structure is same as tree level, but with small corrections

see also arc variables Bellazzini, Miro, Rattazzi, Riembau & Riva, 2011.00037

Higgs upper bounds

fuller use of unitarity & crossing $\Rightarrow$ two-sided bounds

$\mathrm{Im}\, a_\ell > 0 \Rightarrow$

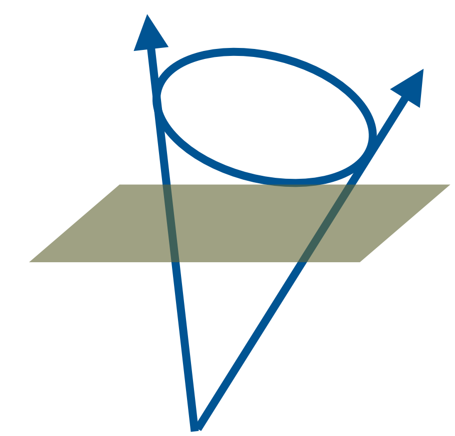


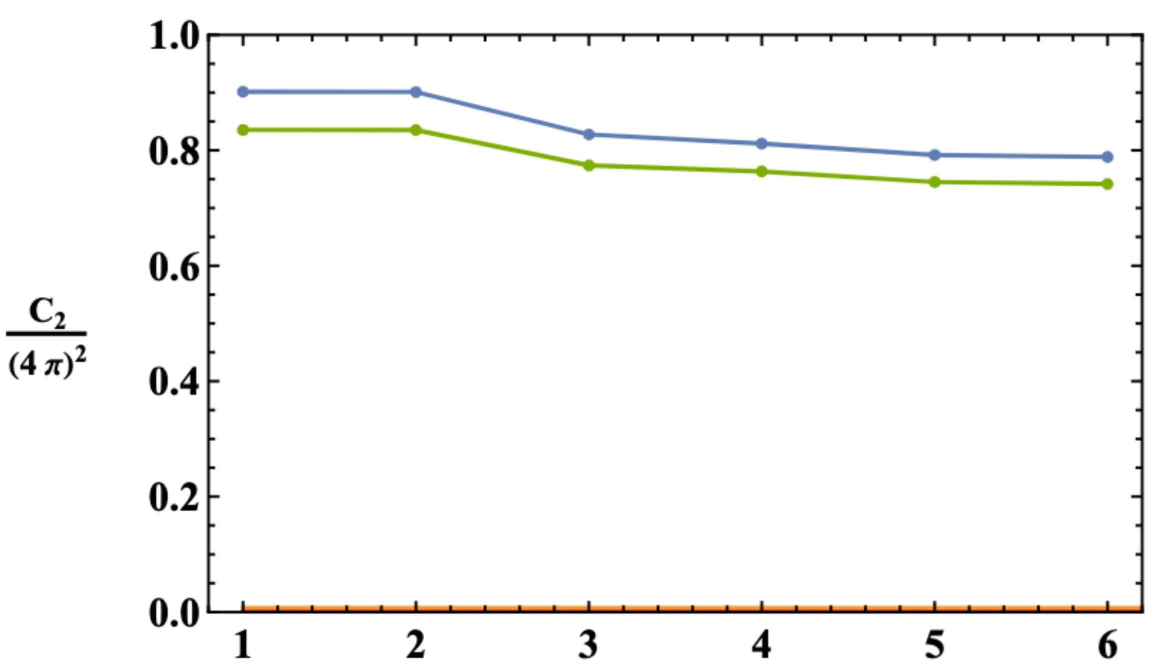
$\mathrm{Im}\, a_\ell > 0$

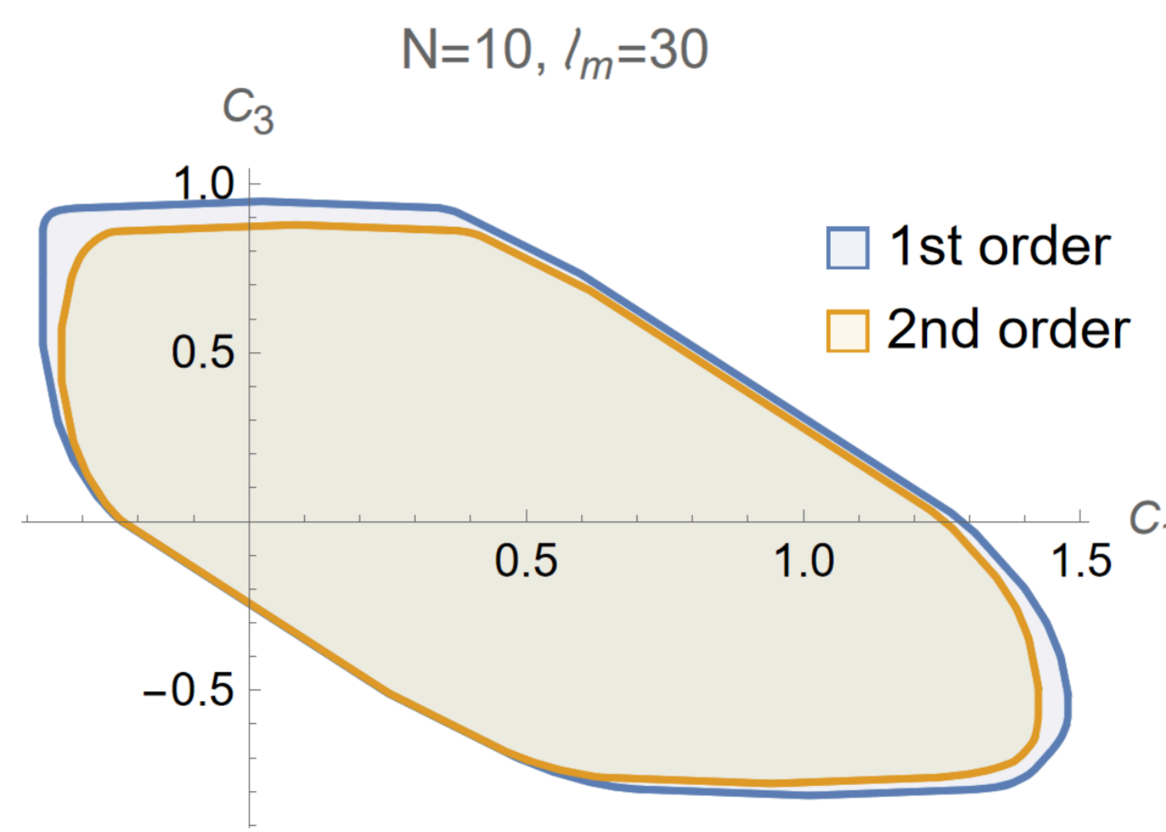
$\mathrm{Im}\, a_\ell < 1$

null constraints

$\Rightarrow$







Chen, Mimasu, Wu, Zhang & **SYZ**, 2206.XXXXX

Reverse bootstrapping

Regge behavior $\lim_{s \to +\infty} \mathrm{Disc}\, \mathcal{A}(s,t) = r(t) s^{\alpha(t)}$

t-channel pole subtracted positivity bound

Planck mass suppressed

$$\partial_s^2 \hat{A}(0,0) > -\beta_s \left(2(\ln r_s)' - (\ln \alpha_s')'\right) + s \leftrightarrow u$$

Consider $h\gamma \to h\gamma$: $\partial_s^2 \hat{A}(0,0) \sim -q_e^2/m_e^2$

1) $(\ln r)' \geq (m_e/q_e)^{-2} \sim \left(10^{-3}\mathrm{GeV}\right)^{-2}$

2) $\left|\ln \alpha'\right|' \sim m_e^2/q_e^2 \Rightarrow$ higer spin $\sqrt{m_e M_s/q_e} \lesssim 10^4 \mathrm{TeV}$

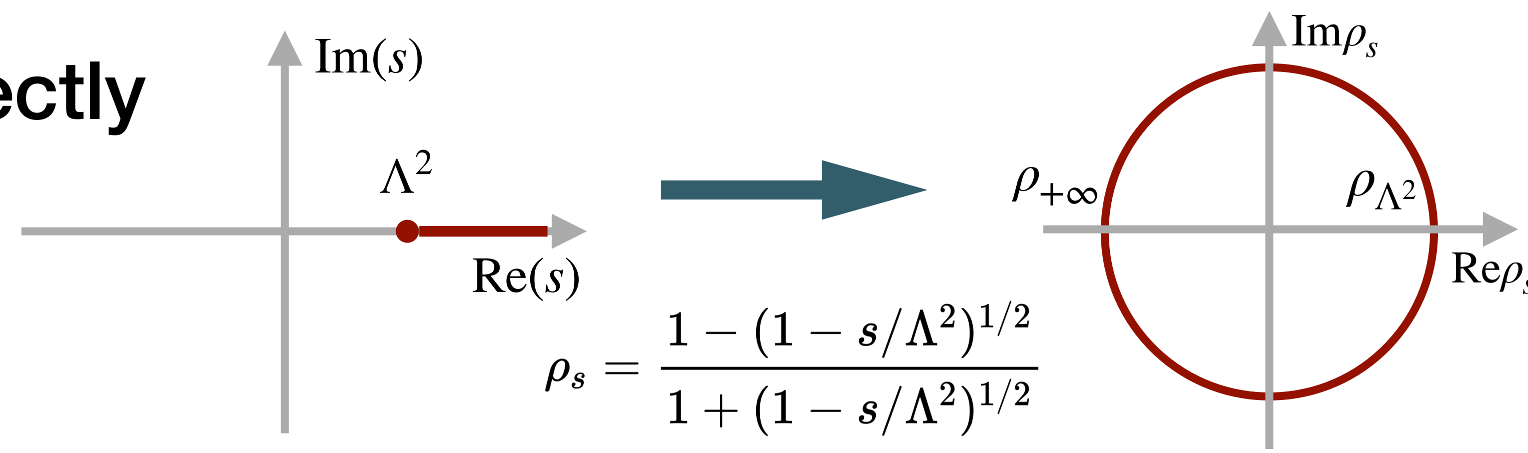
Known IR physics constrains UV physics

Alberte, de Rham, Jaitly & Tolley, PRL, 2111.09226

Primal S-matrix bootstrap

1) Parametrize amplitude directly

$$\mathcal{M}(s,t,u) = \sum_{a,b,c=0} \alpha_{a,b,c}\,\rho_s^a \rho_t^b \rho_u^c$$



$$\rho_s = \frac{1-(1-s/\Lambda^2)^{1/2}}{1+(1-s/\Lambda^2)^{1/2}}$$

analyticity and crossing symmetry are bulit-in

but need to truncate the amplitude expansion

Paulos, Penedones, Toledo, van Rees, Vieira, 1708.06765

2) Impose full unitarity

$$2\,\mathrm{Im}\,A_\ell(s) \geqslant |A_\ell(s)|^2$$

stronger results

numerically less stable

Primal: rule in space **Dispersion relation:** rule out space

EFT of Maxwell (photons): Haring, Hebbar, Karateev, Meineri & Penedones, 2211.05795

Compare primal and dual: Miro, Guerrieri & Gumus, 2210.01502

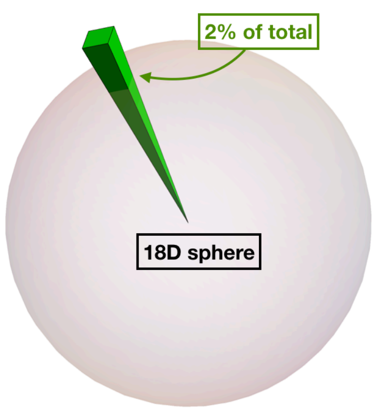
Summary

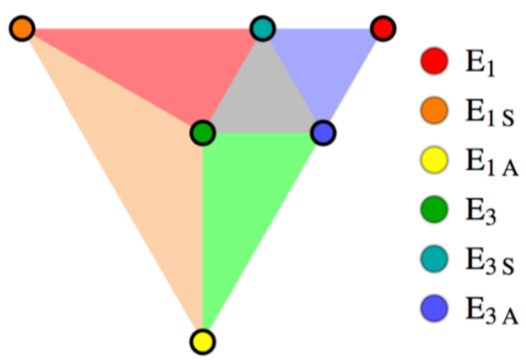
- Positivity bounds are **robust** — from axioms of QFT
- Wilson coefficients of EFT are typically bounded to be:

$$c_i \sim O(1)$$

"naturalness" is a rigorous result!

- Applications in many areas including SMEFT.
- **Significantly constrain** SMEFT
- Venue to test QFT principles
- Important to **reverse engineer UV theory** (thus dim-8)





Thank you!