QCD in 2 Dimensions

An Introduction

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Contents

Overview

Background

Tough problems in the real-world QCD:

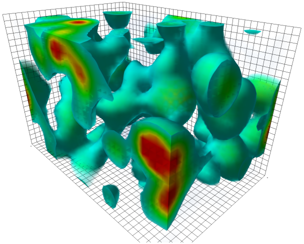
Confinement

Dynamical mass generation

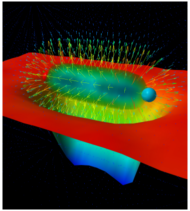
Chiral symmetry breaking

⋮

It would be inspiring if one can find a solvable toy model QFT that reveals some characteristics of the real-world QCD. 't Hooft model (1+1D QCD in the large-N_c limit) is such an ideal theoretical playground, which possesses:
Color confinement, chiral symmetry spontaneous breaking, Regge trajectory, ...



(a) Vacuum structure (CSSM)

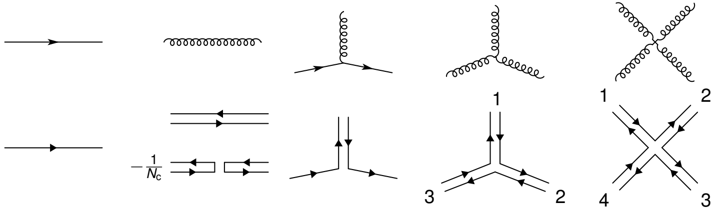


(b) Flux tube (CSSM)

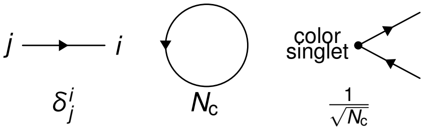
Large-N_c Limit

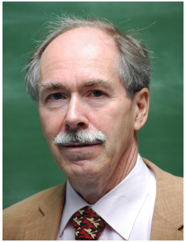
Double-line notation for SU(N_c) gauge field ('t Hooft 1974a)

$$(T^a)^i{}_j\,(T^a)^k{}_l = \frac{1}{2}\left(\delta^i_l\delta^k_j - \frac{1}{N_c}\delta^i_j\delta^k_l\right)$$



N_c counting rules:



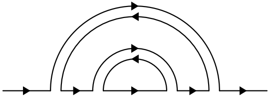


$$V - E + F \equiv \chi, \quad \sum_n V_n \equiv V, \quad \sum_n n V_n = 2E$$

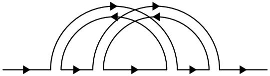
χ: Euler characteristic. Each diagram counts as $\mathcal{O}(r)$:

$$r = g_s^{V_3 + 2V_4} N_c^F = \left(g_s^2 N_c\right)^{\frac{1}{2} V_3 + V_4} N_c^{\chi}$$

$r =$



$g_s^4 N_c^2$



$g_s^4 N_c^0$

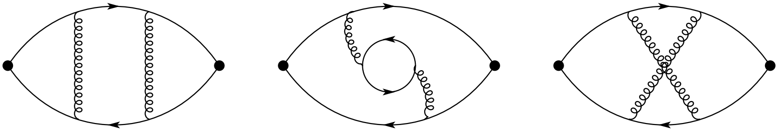
Large N_c limit

$$N_c \to \infty, \quad g_s \to 0, \quad g_s^2 N_c \equiv 4\pi\lambda \quad \text{finite const.}$$

Only planar diagrams play a leading role.

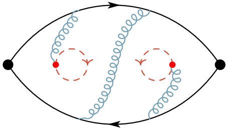
$$r = (4\pi\lambda)^{\frac{1}{2}V_3 + V_4}\, N_c^{\chi} \sim \frac{1}{N_c^{-\chi}}, \quad \chi = 2 - 2g - b$$

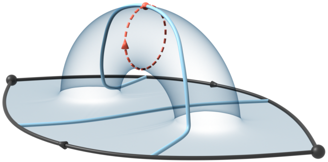
g: genus, b: boundary.



(a) $g = 0,\ b = 1,\ \chi = 1$ (b) $g = 0,\ b = 2,\ \chi = 0$ (c) $g = 1,\ b = 1,\ \chi = -1$

Diagram (c) is non-planar:





Light-Cone(Front) Coordinates

$$x^{\pm} = \frac{x^0 \pm x^z}{\sqrt{2}}, \quad p \cdot x = p^+ x^- + p^- x^+$$

Mass shell:

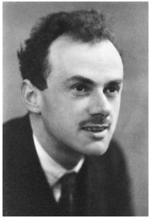
$$p^2 = 2p^+ p^- = m^2$$

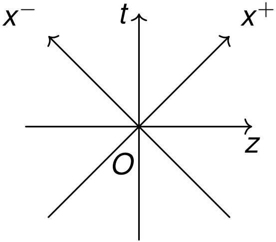
It's natural to utilize the light-cone gauge

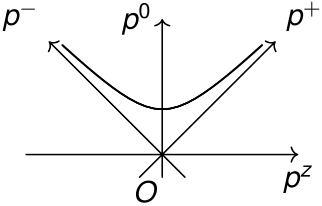
$$A^+ = 0$$

in the light-cone coordinates. In this gauge:

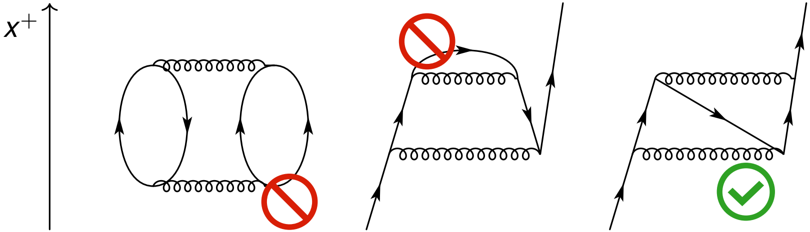
- Non-Abelian gluon self-interactions vanish in 2D
- Gluon exchange between quarks becomes an instant potential







On-shell light-cone momentum $p^+ > 0$,
leading to significant simplification:



(a) (b) (c)

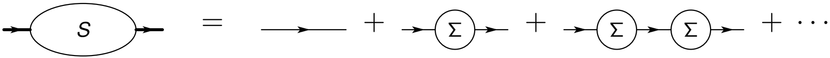
(a) Light-cone vacuum is trivial (full vacuum = perturbative vacuum)

(b) Rainbow diagrams only receive one-loop corrections

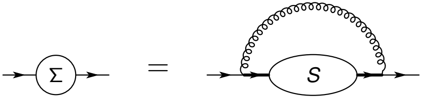
Diagrammatic Approach to the ’t Hooft Model

’t Hooft Model in the Light Cone

Dressed quark propagator



(a)



(b)

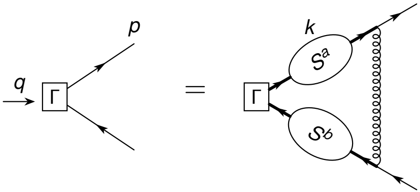
$$S(p) = \frac{2ip^+}{2p^+p^- - m^2 - 2p^+\Sigma(p) + i\epsilon}$$

$$-i\Sigma(p) = \frac{-ig_s^2 N_c}{2} \int \frac{d^2k}{(2\pi)^2} \frac{S(k)}{(p^+ - k^+)^2}$$

$$S(p) = \frac{2ip^+}{2p^+p^- - m^2 + 2\lambda + i\epsilon}$$

Gauge dependent!

Bound state equation

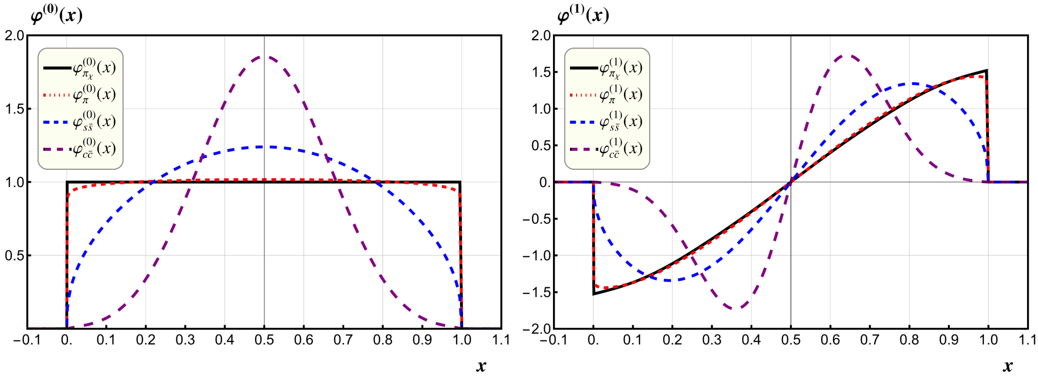


$$\Gamma^{ab}(p;q) = \frac{-ig_s^2 N_c}{2} \int \frac{d^2k}{(2\pi)^2} \frac{1}{(p^+ - k^+)^2} S^a(k)\,\Gamma^{ab}(k;q)\,S^b(k-q)$$

$$\varphi_n^{a\bar{b}}(x) \equiv \frac{2iq^+\Gamma^{ab}(xq^+;q)}{\mu_n^2 - \frac{m_a^2-2\lambda}{x} - \frac{m_b^2-2\lambda}{1-x} + i\epsilon}, \quad 0 < x < 1$$

satisfies the celebrated 't Hooft equation ('t Hooft 1974b):

$$\left(\mu_n^2 - \frac{m_a^2 - 2\lambda}{x} - \frac{m_b^2 - 2\lambda}{1-x}\right)\varphi_n^{a\bar{b}}(x) = -2\lambda \fint_0^1 dy \frac{\varphi_n^{a\bar{b}}(y)}{(x-y)^2}$$



From Jia et al. 2017

Asymptotic Behavior at Boundaries

$$\varphi_n^{a\bar{b}}(x) \propto x^{\beta_a}(1-x)^{\beta_b}, \quad x \to 0, 1$$

where $(q = a, b)$

$$\pi\beta_q \cot \pi\beta_q = 1 - \frac{m_q^2}{2\lambda}$$

In the chiral limit $m_a \sim m_b \to 0$,

$$\beta_q \approx \sqrt{\frac{3}{2\pi^2\lambda}} m_q$$

$$\mu_n^2 \int_0^1 dx \varphi_n^{a\bar{b}}(x) = \int_0^1 dx \left(\frac{m_a^2}{x} + \frac{m_b^2}{1-x} \right) \varphi_n^{a\bar{b}}(x)$$

$$\mu_\pi^2 \approx \sqrt{\frac{2\pi^2\lambda}{3}} (m_a + m_b) \qquad \text{("GOR relation")}$$

’t Hooft Model in Equal Time

It’s natrual to use the axial gauge $A^z = 0$ in equal time.

Dressed quark propagator

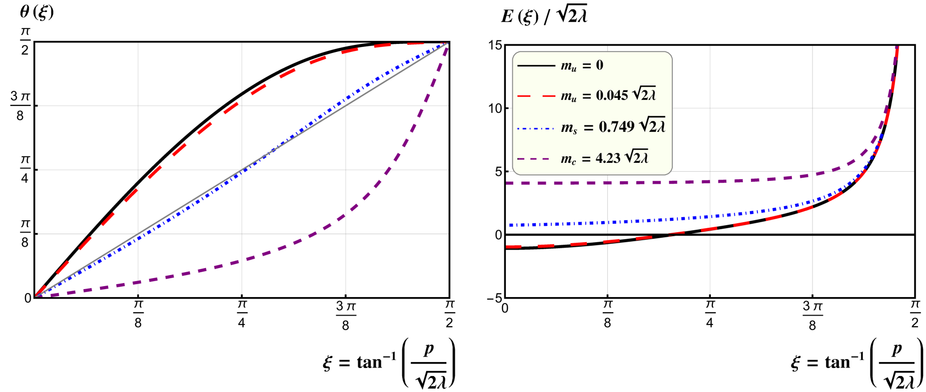
$$S(p) = \frac{i}{\not{p} - m - \Sigma(p)}$$

$$-i\Sigma(p) = \frac{-ig_s^2 N_c}{2} \int \frac{d^2k}{(2\pi)^2} \frac{\gamma^0 S(k) \gamma^0}{(p^z - k^z)^2}$$

$$S(p) = \frac{i}{2} \left(\frac{\gamma^0 + e^{-\theta(p^z)\gamma^z}}{p^0 - E(p^z) + i\epsilon} + \frac{\gamma^0 - e^{-\theta(p^z)\gamma^z}}{p^0 + E(p^z) - i\epsilon} \right)$$

$$E(p^z) \cos\theta(p^z) - m = \frac{\lambda}{2} \int \frac{dk^z}{(p^z - k^z)^2} \cos\theta(k^z)$$

$$E(p^z) \sin\theta(p^z) - p^z = \frac{\lambda}{2} \int \frac{dk^z}{(p^z - k^z)^2} \sin\theta(k^z)$$



$\theta(\xi)$
$\frac{\pi}{2}$
$\frac{3\pi}{8}$
$\frac{\pi}{4}$
$\frac{\pi}{8}$
0
$\frac{\pi}{8}$
$\frac{\pi}{4}$
$\frac{3\pi}{8}$
$\frac{\pi}{2}$
$\xi = \tan^{-1}\left(\frac{p}{\sqrt{2\lambda}}\right)$
$E(\xi)/\sqrt{2\lambda}$
15
10
5
0
−5
$m_u = 0$
$m_u = 0.045\sqrt{2\lambda}$
$m_s = 0.749\sqrt{2\lambda}$
$m_c = 4.23\sqrt{2\lambda}$
0
$\frac{\pi}{8}$
$\frac{\pi}{4}$
$\frac{3\pi}{8}$
$\frac{\pi}{2}$
$\xi = \tan^{-1}\left(\frac{p}{\sqrt{2\lambda}}\right)$

Bound state equation

$$\Gamma^{ab}(p;q) = \frac{-ig_s^2 N_c}{2} \int \frac{d^2k}{(2\pi)^2} \frac{1}{(k^z - p^z)^2} \gamma^0 S^a(k) \Gamma^{ab}(k;q) S^b(k-q) \gamma^0 .$$

$$\varphi_n^{a\bar{b}+}(k^z;q^z) \equiv -\frac{-i\bar{u}^a(k^z)\Gamma^{ab}(k^z;q)v^b(q^z-k^z)}{q_n^0 - E^b(k^z-q^z) - E^a(k^z) + i\epsilon}$$

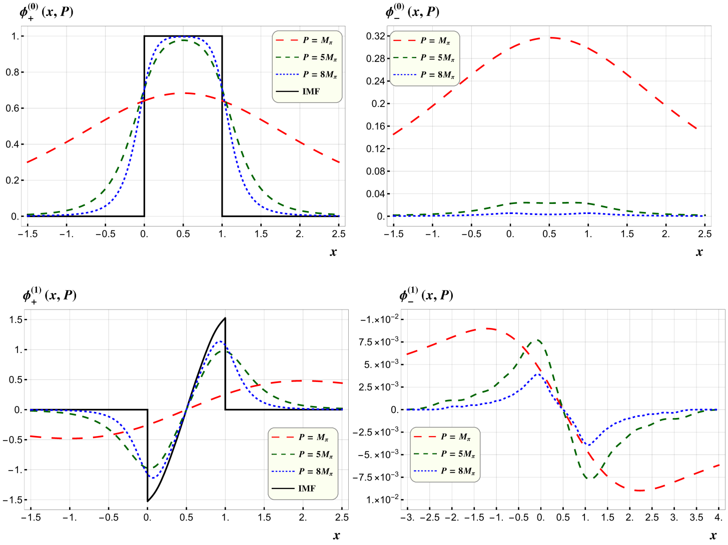
$$\varphi_n^{a\bar{b}-}(k^z;q^z) \equiv \frac{-i\bar{v}^a(-k^z)\Gamma^{ab}(k^z;q)u^b(k^z-q^z)}{-q_n^0 - E^b(k^z-q^z) - E^a(k^z) + i\epsilon}$$

Bars-Green equations (Bars and Green 1978):

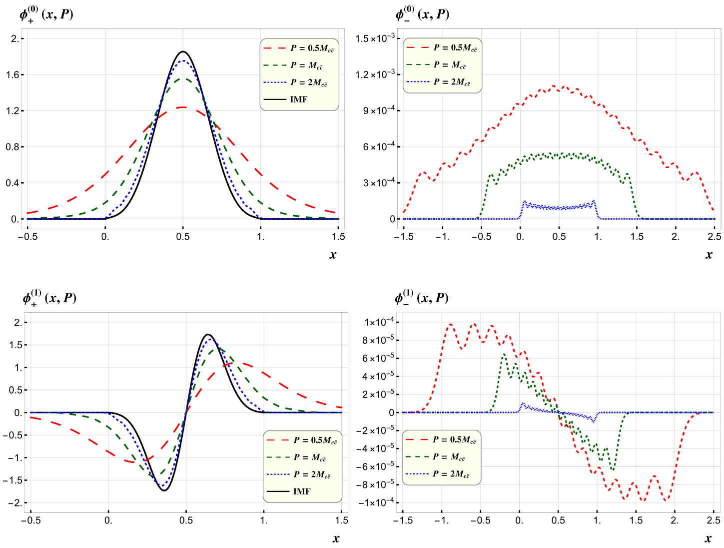
$$\left[\pm q_n^0 - E^b(p^z-q^z) - E^a(p^z)\right] \varphi_n^{a\bar{b}\pm}(p^z;q^z) = -\lambda \int \frac{dk^z}{(k^z-p^z)^2}$$
$$\times \left[C^{ab}(p^z,k^z;q^z)\varphi_n^{a\bar{b}\pm}(k^z;q^z) - S^{ab}(p^z,k^z;q^z)\varphi_n^{a\bar{b}\mp}(k^z;q^z)\right],$$

Comparison between ET & LC

Chiral limit:



Heavy quark:



$\phi_+^{(0)}(x, P)$
$\phi_-^{(0)}(x, P)$
$\phi_+^{(1)}(x, P)$
$\phi_-^{(1)}(x, P)$
$P = 0.5M_{c\bar{c}}$
$P = M_{c\bar{c}}$
$P = 2M_{c\bar{c}}$
IMF
x

Chiral Condensate

Coleman's No-Go Theorem

Theorem (Coleman 1973)

There is no Goldstone boson in 2 dimensions.

However, it was first pointed out in Witten 1978 that, in the large-N_c limit, the chiral symmetry may be "almost" spontaneously broken due to the Berezinski-Kosterfitz-Thouless effect, where the correlator

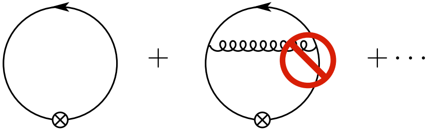
$$\langle 0|\bar{\psi}\psi(x)\,\bar{\psi}\psi(0)\,|0\rangle \sim |x|^{-1/N_c}$$

shows long-range order.

Quark Condensate – Light-Cone Calculation

$$\langle 0|\bar\psi\psi_{\text{ren}}|0\rangle = \langle 0|\bar\psi\psi|0\rangle - \langle 0|\bar\psi\psi_{\text{free}}|0\rangle$$

A naive attempt



gives $\langle 0|\bar\psi\psi|0\rangle = \langle 0|\bar\psi\psi_{\text{free}}|0\rangle \to 0 \;\; (m_q \to 0)$. It fails to produce the correct condensate.

The correct way is to consider the pseudoscalar-axial-vector correlator (Zhitnitsky 1985; Burkardt 1994):

$$\int d^2x \; e^{iqx} \; \partial_\mu \langle 0|\mathrm{T}\bar\psi\gamma^\mu\gamma_5\psi\,(x)\;\; \bar\psi\gamma_5\psi\,(0)\,|0\rangle$$

Method 1 (OPE, Zhitnitsky 1985):

$$
\begin{aligned}
2\langle 0|\bar{\psi}\psi|0\rangle &\overset{\text{OPE}}{\Leftarrow} \lim_{q^2\to-\infty}\int d^2x\, e^{iqx}\, \partial_\mu\langle 0|\mathrm{T}\bar{\psi}\gamma^\mu\gamma_5\psi(x)\ \bar{\psi}\gamma_5\psi(0)\,|0\rangle \\
&= -i\sum_n \langle 0|\bar{\psi}\not{q}\gamma_5\psi|n\rangle \frac{i}{q^2}\langle n|\bar{\psi}\gamma_5\psi|0\rangle \\
&= -AB
\end{aligned}
$$

where only $n=\pi$ contributes and π possesses an off-shell (analytic continuated) momentum q:

$$
\begin{aligned}
A &\equiv \frac{1}{q^2}\langle 0|\bar{\psi}\not{q}\gamma_5\psi|\pi\rangle = \sqrt{\frac{N_c}{\pi}}\int_0^1 dx\varphi_\pi(x) \to \sqrt{\frac{N_c}{\pi}} \\
B &\equiv \langle 0|\bar{\psi}\gamma_5\psi|\pi\rangle = \frac{m_q}{2}\sqrt{\frac{N_c}{\pi}}\int_0^1 dx\frac{\varphi_\pi(x)}{x(1-x)} \to \sqrt{\frac{2\pi N_c\lambda}{3}}
\end{aligned}
$$

Method 2 (Schwinger-Dyson Eq., Burkardt 1994):

$$0 = \lim_{q^\mu \to 0} \int d^2x\, e^{iqx}\, \partial_\mu \langle 0 | T \bar{\psi}\gamma^\mu\gamma_5\psi(x)\ \bar{\psi}\gamma_5\psi(0) | 0 \rangle$$
$$= 2\langle 0|\bar{\psi}\psi|0\rangle + 2im_q \int d^2x \langle 0 | T\bar{\psi}\gamma_5\psi(x)\ \bar{\psi}\gamma_5\psi(0)|0\rangle$$

$$\langle 0|\bar{\psi}\psi|0\rangle = -im_q \sum_n \frac{i}{-\mu_n^2} f_n^2 \to -\frac{m_q}{\mu_\pi^2} B^2, \quad m_q \to 0$$

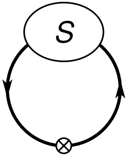
where $f_n \equiv \langle 0|\bar{\psi}\gamma_5\psi|n\rangle$.

$$\langle 0|\bar{\psi}\psi|0\rangle = -N_c \sqrt{\frac{\lambda}{6}}$$

Quark Condensate – Equal-Time Calculation

Fairly straightforward ($m_q \to 0$, $\lambda = 1/2$, Li 1986):

$$\langle \Omega | \bar{\psi}\psi | \Omega \rangle =$$



$$= -N_c \int \frac{dp^z}{2\pi} \cos\theta\,(p^z) = -0.29 N_c$$

Numerically consistent with the light-cone result!

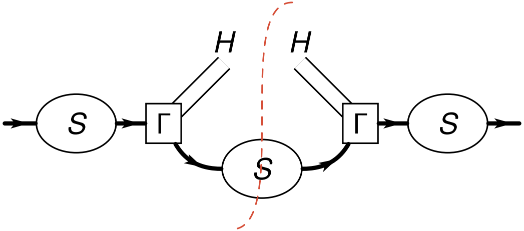
Quark Fragmentation Function

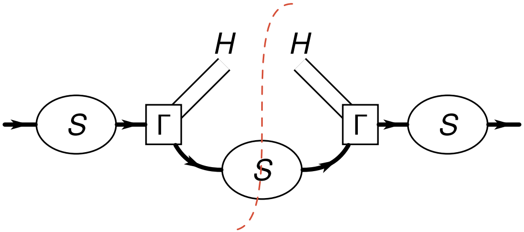
Rigorous Result

The Collins-Soper definition of FF is given by

$$D_{q\to H}(z) = \sum_X \frac{1}{4\pi z}\int dx^- e^{-iP^+x^-/z}\frac{1}{N_c}\mathrm{tr}_{\mathrm{color}}\mathrm{tr}_{\mathrm{Dirac}}$$
$$\times\left\{(n\cdot\gamma)\langle 0|\bar{\mathrm{T}}W[\infty,0]\psi(0,0)|H,X,\mathrm{out}\rangle\right.$$
$$\left.\times\langle H,X,\mathrm{out}|\mathrm{T}\bar{\psi}(0,x^-)\,W[\infty,x^-]^\dagger|0\rangle\right\}.$$

In the light-cone gauge, the light-like Wilson line $W=1$.





Generally for all x ($q^+ > 0$),

$$\Gamma^{ab}\left(xq^+; q\right) = \frac{i\lambda}{q^+}\int_0^1 dy \frac{\varphi_n^{a\bar{b}}(y)}{(x-y)^2}$$

Rigorous result:

$$D_{q\to H}(z) = \frac{4\lambda^2}{N_c}\frac{(1-z)^2}{z\left[z^2(m_q^2-2\lambda)+(1-z)M_H^2\right]^2}\left[\int_0^1 dy\,\frac{\varphi_n(y)}{(y-1/z)^2}\right]^2$$

NRQCD Factorization in the Heavy Quark Limit

In the heavy-quark limit, fragmentation functions can be factorized as

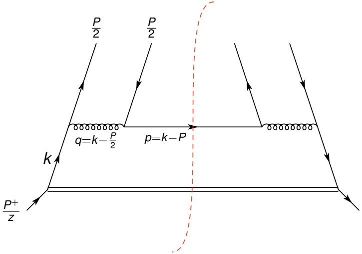
$$D_{q\to H}(z) = d_1(z)\langle 0|\mathcal{O}_1^H|0\rangle + \mathcal{O}\left(v^2\right).$$

with the NRQCD production operator

$$\mathcal{O}_1^H = \chi^\dagger\psi\sum_X |H+X\rangle\langle H+X|\psi^\dagger\chi$$

whose VEV can be approximated by

$$\left\langle\mathcal{O}_1^H\right\rangle \approx {}_{\mathrm{NR}}\langle H|\psi^\dagger\chi|0\rangle\langle 0|\chi^\dagger\psi|H\rangle_{\mathrm{NR}}$$



Perturbative matching calculation gives

$$D_{q\to H}(z) = \frac{64\pi\lambda^2}{N_c^2}\frac{(1-z)^2 z^3}{(2-z)^8 m_q^5}\left\langle\mathcal{O}_1^H\right\rangle$$

Comparison Between Rigorous & NRQCD Results

In the heavy quark limit, the ground state wave function

$$\varphi_0(x) = c\delta\left(x - \frac{1}{2}\right)$$

where the normalization constant c is related to the decay constant:

$$c = \int_0^1 dx\, \varphi_0(x) = \sqrt{\frac{\pi}{N_c}}\frac{1}{P^0}\langle\Omega|\bar{\psi}\gamma^0\gamma_5\psi|H(P^z=0)\rangle$$

$$= \sqrt{\frac{\pi}{N_c}}\frac{\sqrt{2M_H}}{M_H}\langle 0|\chi^\dagger\psi|H\rangle_{\mathrm{NR}}$$

The rigorous result is approximated by

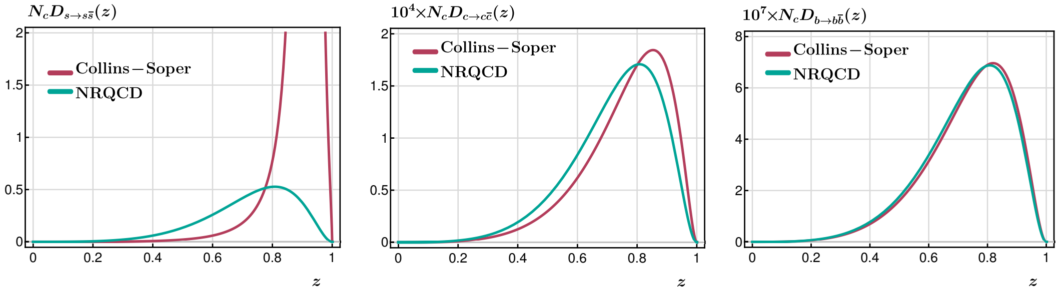
$$D_{q\to H}(z) \approx \frac{4\lambda^2}{N_c}\frac{(1-z)^2}{z(z-2)^4\, m_q^4}\left[\frac{1}{(1/2-1/z)^2}\right]^2 |c|^2$$

which is exactly of the form predicted by NRQCD factorization.

Numerical results

$$\langle 0|\chi^\dagger \psi|H\rangle_{\text{NR}} = \sqrt{N_c}\psi_H(0)$$

The wave function at origin $\psi_H(0)$ is derived from the Schrödinger equation with linear potential.



TBC

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