Using AMFlow to calculate integrals

——dimensional regularization/lattice regularization

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强子质量的非微扰起源，中科院理论所，2023/04/17

北京大學

Outline

I. Introduction

II. Linear space of Feynman integrals

III. Auxiliary mass flow method

- **1) Kinematics**
- **2) Spacetime and loops**
- **3) Phenomenology**

IV. Current status and outlook

Precision: gateway to discovery

➢ Positive results

- Rudolphine Tables (Tycho Brahe's data, most precise before telescope): Kepler's laws, Newton's law of gravity
- Accurate black-body radiation data: Planck's quantization
- The advance of the Mercury's perihelion: hint of Einstein's General Relativity

➢ Negative results

- Michelson's experiments: Einstein's Special Relativity
- No evidence of FCNC: GIM mechanism predicting charm quark, discovery of J/ψ (Ting & Richter)

➢ Discovery via precision

- Search anomalous deviations from theory
- Interplay between exp. and th.
- Theory: lattice or perturbative

Perturbative QFT

1. Generate Feynman amplitudes

- Feynman diagrams and Feynman rules
- New developments: unitarity, recurrence relation, CHY, …

2. Calculate Feynman loop integrals (FIs)

- Amplitudes: linear combinations of FIs with rational coefficients

3. Perform phase-space integrations

- Monte Carlo simulation with IR subtractions
- Relating to loop integrals via reverse unitarity (if no jet)

$$\int \frac{\mathrm{d}^D p}{(2\pi)^D}(2\pi)\delta_+(p^2) = \int \frac{\mathrm{d}^D p}{(2\pi)^D}\left(\frac{\mathrm{i}}{p^2+\mathrm{i}0^+} + \frac{-\mathrm{i}}{p^2-\mathrm{i}0^+}\right)$$

Definition of FIs

➢ A family of Feynman integrals

$$I_{\vec{\nu}}(D,\vec{s}) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{(\mathcal{D}_1 + \mathrm{i}0^+)^{\nu_1} \cdots (\mathcal{D}_K + \mathrm{i}0^+)^{\nu_K}}$$

$$\mathcal{D}_\alpha = A_{\alpha ij}\ell_i \cdot \ell_j + B_{\alpha ij}\ell_i \cdot p_j + C_\alpha$$

- $\ell_1, \dots, \ell_L$: loop momenta; $p_1, \dots, p_E$: external momenta;
- A, B: integers; C: linear combination of $\vec{s}$ (including masses)
- $\mathcal{D}_1, \dots, \mathcal{D}_K$: inverse propagators; $\nu_1, \dots, \nu_K$: integers
- $\mathcal{D}_{K+1}, \dots, \mathcal{D}_N$: to form complete bases; $\nu_{K+1}, \dots, \nu_N$: non-negative integers

➢ Difficulties of calculating FIs

- Analytical: known special functions are insufficient to express FIs
- Numerical: UV, IR, integrable singularities, …

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Integration-by-parts: example

- **A family of FIs:** $F(n)=\int\frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{1}{(\ell^2-\Delta)^n}$

➢ Vanishing on the big hypersphere with radius R

Lagrange, Gauss, Green, Ostrogradski, 1760s-1830s

'‘t Hooft, Veltman, NPB (1972)

$$\int\frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{\partial}{\partial\ell^\mu}\left[\frac{\ell^\mu}{(\ell^2-\Delta)^n}\right]\overset{\Downarrow}{=}\int_\partial\frac{\mathrm{d}^{D-1}S_\mu}{(2\pi)^D}\left[\frac{\ell^\mu}{(\ell^2-\Delta)^n}\right]\overset{\Downarrow}{=}0$$

- **Integrand: fixed power in R; Measure: R^{D-1}**
- **Thus vanishing in dimensional regularization**

➢ Relations between FIs

$$0=\int_\ell\left[\frac{D}{(\ell^2-\Delta)^n}-n\int_\ell\frac{2(\ell^2-\Delta)+2\Delta}{(\ell^2-\Delta)^{n+1}}\right]\quad=(D-2n)F(n)-2n\Delta F(n+1)$$

$$F(n+1)=\frac{1}{-\Delta}\frac{n-\frac{D}{2}}{n}F(n)$$

- **All FIs in this family can be expressed by $F(1)$**

IBP equations

➢ **Dimensional regularization: vanish at boundary**

't Hooft, Veltman, NPB (1972)
Chetyrkin, Tkachov, NPB (1981)

$$\int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\partial}{\partial \ell_j^\mu} \left(q_k^\mu \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{\mathcal{D}_1^{\nu_1} \cdots \mathcal{D}_K^{\nu_K}} \right) = 0, \qquad \forall \vec{\nu}, j, k$$

$\overrightarrow{q^\mu} = (\ell_1^\mu, \cdots, \ell_L^\mu, p_1^\mu, \cdots, p_E^\mu)$

$$\Downarrow$$

- **Linear equation:** $\sum_{\vec{\nu}'} Q_{\vec{\nu}'}^{\vec{\nu} jk}(D, \vec{s}) I_{\vec{\nu}'}(D, \vec{s}) = 0$
- Q: **polynomials in** $D, \vec{s}$
- **Plenty of linear equations can be easily obtained by varying:** $\vec{v}, j, k$

Master integrals

➢ # of equations grows faster than # of FIs

Laporta, Remiddi, 9602417, Gehrmann, Remiddi, 9912329

- Let positive powers $r=\nu_{i_1}+\cdots+\nu_{i_z}$, nonpositive $s=-(\nu_{i_{z+1}}+\cdots+\nu_{i_N})$, $N_{r,s}=C_{r-1}^{z-1}C_{s+N-z-1}^{N-z-1}$ is the # of FIs with fixed r,s
- # of equations (for seeds with fixed r,s) $=L(L+E)\times N_{r,s}$
- # of new FIs $=N_{r+1,s}+N_{r+1,s+1}$ ($\approx 2\,N_{r,s}$ for sufficient large r,s)
- Expectation: finite # of linearly independent FIs

➢ A family of FIs form a FINITE-dim. linear space

Proved by: Smirnov, Petukhov, 1004.4199

- Bases of the linear space called master integrals (MIs)
- IBPs reduce tens of thousands of FIs to much less MIs

IBP reduction

➢ Laporta's algorithm to do reduction

$$\sum_{\vec{\nu}'} Q_{\vec{\nu}'}^{\vec{\nu}jk}(D,\vec{s})I_{\vec{\nu}'}(D,\vec{s}) = 0$$

Laporta, 0102033

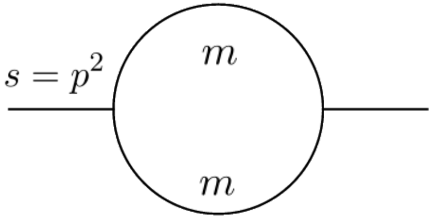
- Generate eqs for all $\vec{\nu}$ with $r \in [r_{\mathrm{min}}, r_{\mathrm{max}}],\ s \in [s_{\mathrm{min}}, s_{\mathrm{max}}]$
- Ordering: simpler FI has smaller z, then smaller r, then smaller s
- Solving linear eqs to eliminate more complicated FIs
- Eventually, all FIs are linear combinations of MIs

➢ Solving IBP eqs: automatic, any-loop order

- Public codes: AIR, FIRE, LiteRed, Reduze, Kira, FiniteFlow, Blade…
- Many more private codes
- Warning: time-consuming for complicated problems

Differential equations: example

➢ Due to IBP: DEs of MIs



$$I_{\nu_1\nu_2} = \int \frac{\mathrm{d}^D\ell}{\mathrm{i}\pi^{D/2}} \frac{1}{(\ell^2 - m^2)^{\nu_1}[(\ell+p)^2 - m^2]^{\nu_2}}$$

$$\begin{cases} \dfrac{\partial}{\partial m^2} I_{11} = I_{21} + I_{12} \overset{\text{IBP}}{=} \dfrac{2(D-3)}{4m^2 - s} I_{11} - \dfrac{D-2}{m^2(4m^2-s)} I_{10} \\ \dfrac{\partial}{\partial m^2} I_{10} = I_{20} \overset{\text{IBP}}{=} \dfrac{D-2}{2m^2} I_{10} \end{cases}$$

$$\begin{cases} \dfrac{\partial}{\partial s} I_{11} = \dfrac{p^\mu}{2s} \dfrac{\partial}{\partial p^\mu} I_{11} = -\dfrac{1}{2s} \displaystyle\int \frac{\mathrm{d}^D\ell}{\mathrm{i}\pi^{D/2}} \frac{2(\ell+p)\cdot p}{(\ell^2 - m^2)[(\ell+p)^2 - m^2]^2} \\ \qquad\qquad = -\dfrac{sI_{12} + I_{11} - I_{02}}{2s} \overset{\text{IBP}}{=} a_{11} I_{11} + a_{10} I_{10} \\ \dfrac{\partial}{\partial s} I_{10} = 0 \end{cases}$$

➢ Boundary Condition

$$\begin{cases} I_{11}|_{m^2 \to 0} = (-s)^{D/2-2} \Gamma(2 - D/2) \dfrac{\Gamma(D/2-1)^2}{\Gamma(D-2)} \\ I_{10} \end{cases}$$

Traditional DEs method

➢ **Step 1: Set up $\vec{s}$-DEs of MIs**

- Differentiate MIs w.r.t. invariants $\vec{s}$, such as $m^2, p \cdot q$

Kotikov, PLB(1991)

- IBP relations result in: $\frac{\partial}{\partial s_i}\vec{I}(D,\vec{s}) = A_i(D,\vec{s})\vec{I}(D,\vec{s})$

➢ **Step 2: Calculate boundary condition**

- Calculate integrals at special value of m^2, p^2
- Case by case, not systematic, maybe still hard!

➢ **Step 3: Solve DEs**

- Systematic, not hard (explain later)

Since 90s'

Using IBPs: express any FI as linear combination of MIs, also setup DEs for MIs

FIs $\triangleq$ Linear algebra $\oplus$ Master integrals

Input:

The same kinematics

The same spacetime dimension

The same number of loops

DEs method: needs BCs

Outline

Auxiliary mass terms

X. Liu, YQM, C. Y. Wang, 1711.09572

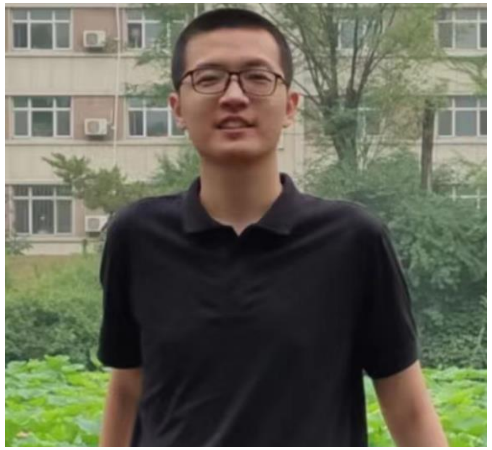
➢ Auxiliary FIs

$$I_{\vec{\nu}}^{\text{aux}}(D,\vec{s},\eta) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{(\mathcal{D}_1 - \lambda_1 \eta + \mathrm{i}0^+)^{\nu_1} \cdots (\mathcal{D}_K - \lambda_K \eta + \mathrm{i}0^+)^{\nu_K}}$$

- $\lambda_i \geq 0$ **(typically** 0 **or** 1**), an auxiliary mass if** $\lambda_i > 0$
- **Analytical function of** η
- **Physical result obtained by (causality)**

$$I_{\vec{\nu}}(D,\vec{s}) \equiv \lim_{\eta \to \mathrm{i}0^-} I_{\vec{\nu}}^{\text{aux}}(D,\vec{s},\eta)$$

- **1) Setup** η**-DEs; 2) Calculate boundary conditions; 3) Solve** η**-DEs**



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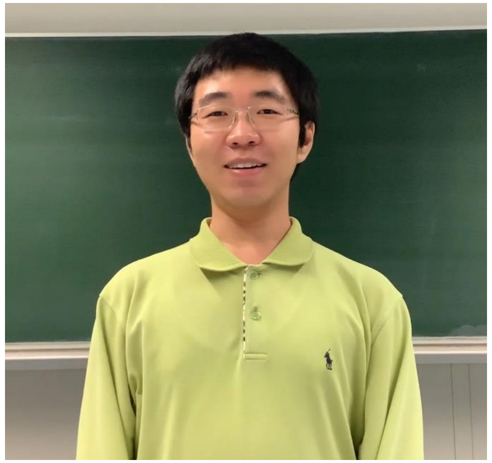


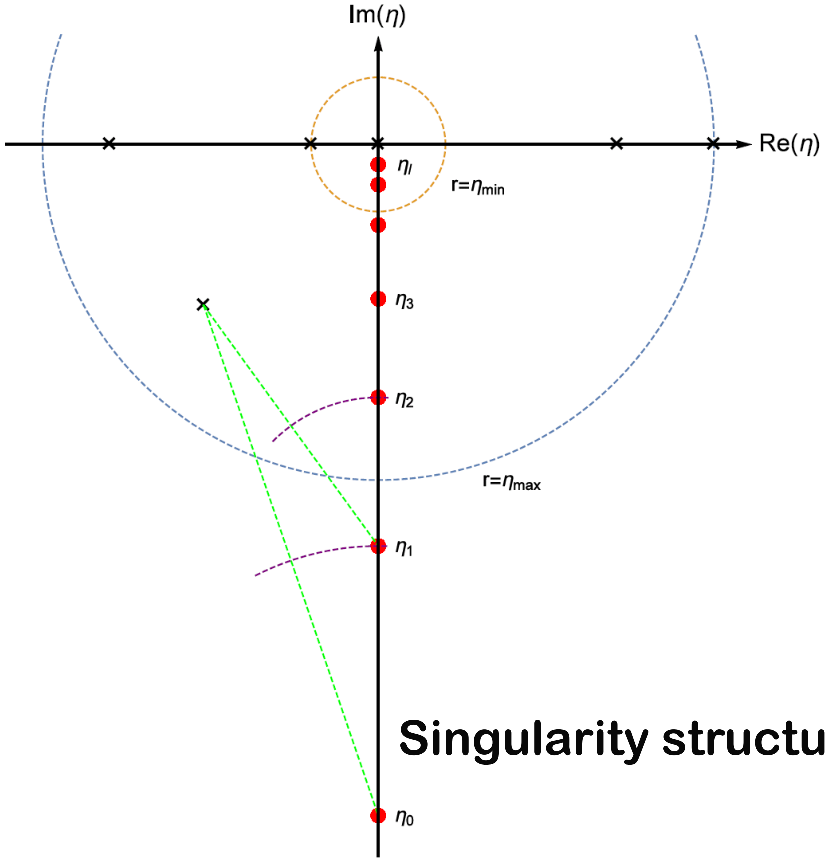
王辰宇，卡尔斯鲁厄大学

➢ η-DEs for MIs in auxiliary family using IBP

$$\frac{\partial}{\partial \eta} \vec{I}^{\text{aux}}(D,\vec{s},\eta) = A(D,\vec{s},\eta) \vec{I}^{\text{aux}}(D,\vec{s},\eta)$$

Flow of auxiliary mass

➢ Solve ODEs of MIs



Singularity structure

$$\frac{\partial}{\partial \eta}\vec{I}^{\text{aux}}(D,\vec{s},\eta) = A(D,\vec{s},\eta)\vec{I}^{\text{aux}}(D,\vec{s},\eta)$$

- If $\vec{I}^{aux}(D,\vec{s},\infty)$ is known , solving ODEs numerically to obtain $\vec{I}^{aux}(D,\vec{s},\mathrm{i}0^-)$
- A well-studied mathematical problem

Step1: Asymptotic expansion at $\eta = \infty$
Step2: Taylor expansion at analytical points
Step3: Asymptotic expansion at $\eta = 0$

- Efficient to get high precision :
  ODEs, known singularity structure

Boundary values at $\eta \to \infty$

➢ Nonzero integration regions as $\eta \to \infty$

- Linear combinations of loop momenta: $\mathcal{O}(\sqrt{|\eta|})$ or $\mathcal{O}(1)$

Beneke, Smirnov, 9711391
Smirnov, 9907471

➢ Simplify propagators at $\eta \to \infty$

- ℓ_L is the $\mathcal{O}(\sqrt{|\eta|})$ part of loop momenta
- ℓ_S is the $\mathcal{O}(1)$ part of loop momenta
- p is linear combination of external momenta

$$\frac{1}{(\ell_\mathrm{L}+\ell_\mathrm{S}+p)^2-m^2-\kappa\,\eta} \sim \frac{1}{\ell_\mathrm{L}^2-\kappa\,\eta}$$

- Unchange if $\ell_L = 0$ and $\kappa = 0$

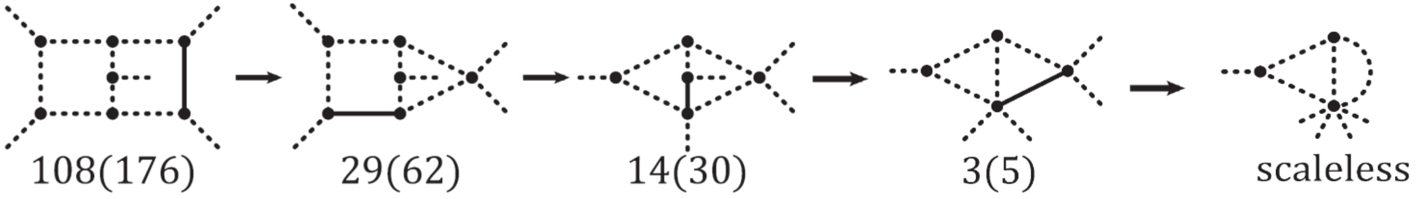
➢ Boundary FIs after simplification

1. Vacuum integrals
2. Simpler FIs with less denominators, if all loop momenta are $\mathcal{O}(1)$

Iterative strategy

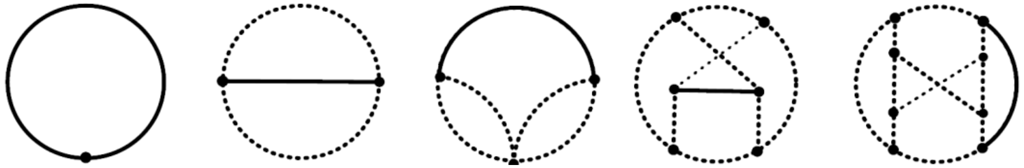
- **For boundary FIs with less denominators:**
  - Calculate them again use AMF method, even simpler boundary FIs as input (besides vacuum integrals)

X. Liu, YQM, 2107.01864



  - Eventually, leaving only (single-mass) vacuum integrals as input
  - Kinematic information can be recovered by linear algebra!

- **Typical single-mass vacuum MIs**



Baikov, Chetyrkin, 1004.1153
Lee, Smirnov, Smirnov, 1108.0732
Georgoudis, et. al., 2104.08272

  - Much simpler to be calculated
  - The same number of loops and spacetime dimensions

2017-2021

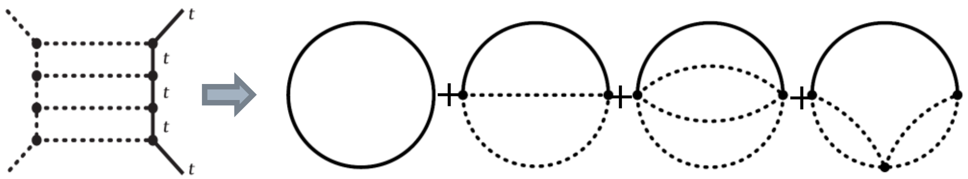
FIs ≜ Linear algebra ⊕ Vacuum integrals

Input:

No kinematics (no external legs)

The same spacetime dimension

The same number of loops



Is this the end of the story?

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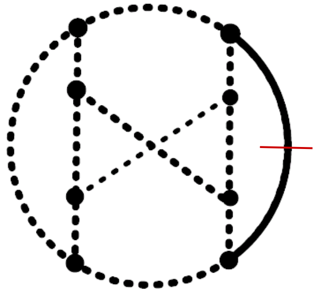
From vacuum integrals to p-integrals

➢ A family of single-mass vacuum integrals

$$I_{\vec{\nu}}(D, m^2) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{(\mathcal{D}_1 + \mathrm{i}0^+)^{\nu_1} \cdots (\mathcal{D}_K + \mathrm{i}0^+)^{\nu_K}}$$

$$\mathcal{D}_1 = \ell_1^2 - m^2 + \mathrm{i}0^+$$



- m^2: the only scale. Can choose $m^2 = 1$

➢ Propagator (p-)integrals

$$\widehat{I}_{\vec{\nu}'}(\ell_1^2) = \int \left(\prod_{i=2}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \right) \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{\mathcal{D}_2^{\nu_2} \cdots \mathcal{D}_K^{\nu_K}}$$

$$\vec{\nu}' = (\nu_2, \cdots, \nu_N)$$
$$\nu = \sum_{i=1}^{N} \nu_i$$

- As ℓ_1^2 is the only scale: $\widehat{I}_{\vec{\nu}'}(\ell_1^2) = (-\ell_1^2)^{\frac{(L-1)D}{2} - \nu + \nu_1} \widehat{I}_{\vec{\nu}'}(-1)$
- L-loop single-mass vacuum integral expressed by $(L-1)$-loop p-integral

$$I_{\vec{\nu}} = \int \frac{\mathrm{d}^D \ell_1}{\mathrm{i}\pi^{D/2}} \frac{(-\ell_1^2)^{\frac{(L-1)D}{2} - \nu + \nu_1}}{(\ell_1^2 - 1 + \mathrm{i}0^+)^{\nu_1}} \widehat{I}_{\vec{\nu}'}(-1) = \frac{\Gamma(\nu - LD/2)\Gamma(LD/2 - \nu + \nu_1)}{(-1)^{\nu_1}\Gamma(\nu_1)\Gamma(D/2)} \widehat{I}_{\vec{\nu}'}(-1)$$

From p-integrals to vacuum integrals

Z. F. Liu, YQM, 2201.11637

➢ Apply AMF method on $(L-1)$-loop p-integral

1) IBP to setup η-DEs
2) Single-mass vacuum integrals no more than $(L-1)$ loops as input

Single-mass vacuum integrals with L loops are determined by that with no more than $(L-1)$ loops (besides IBP)

- Boundary: 0-loop p-integrals equal 1

刘志峰，浙江大学

➢ Only IBPs are needed to determine FIs!

- IBPs: linear algebra, exact (in $D, \vec{s}$) relations between FIs

Workflow

- The 'FICalc' to calculate FIs can be defined as (any given nonsingular D and $\vec{s}$):

  ① If it is a 0-loop p-integral, return 1;

  ② If it is a single-mass vacuum integral, express it by a p-integral, and call 'FICalc' to calculate the p-integral;

  ③ Otherwise:

  a) Introduce η to one propagator (if the mass mode is not possible)

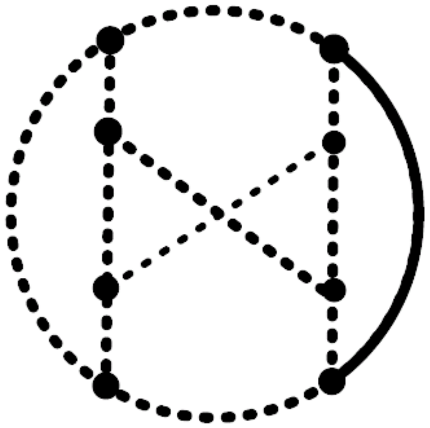
  b) Setup η-DEs using IBP as input

  c) Call 'FICalc' to calculate boundary FIs at $\eta \to \infty$

  d) Numerically solve η-DEs with given BCs to obtain $\eta \to \mathrm{i}0^-$

A five-loop example

Z. F. Liu, YQM, 2201.11637



$$= \begin{aligned} &- 2.073855510286740\epsilon^{-2} - 7.812755312590133\epsilon^{-1} \\ &- 17.25882864945875 + 717.6808845492140\epsilon \\ &+ 8190.876448160049\epsilon^{2} + 78840.29598046500\epsilon^{3} \\ &+ 566649.1116484678\epsilon^{4} + 3901713.802716081\epsilon^{5} \\ &+ 23702384.71086095\epsilon^{6} + 142142936.8205112\epsilon^{7}, \end{aligned}$$

- IBP relations are the only input!
- Terms up to $\mathcal{O}(\epsilon^4)$ agree with literature; Others are new ($D = 4 - 2\epsilon$)

  Lee, Smirnov, Smirnov, 1108.0732
- An arbitrary dimension D = 4/7, challenging for other methods

-9.79311209704864932180879598006911164642818254746542833061469472644315160318306100566682423418773094010322939010045743194940172060911582447082246541938856806619503723720902111961684999664025920163632 1*10^7

with about 130 significant digits

Since 2022

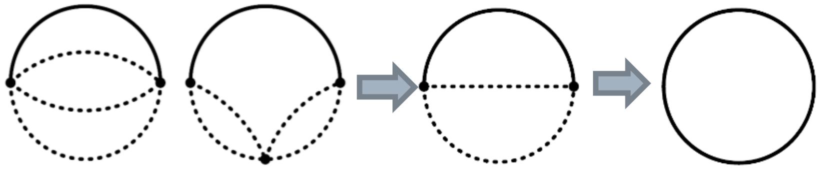
FIs $\triangleq$ Linear algebra

No other input:

No kinematics!

No spacetime dimension!

No loops!



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Package: AMFlow

X. Liu, YQM, 2201.11669

➢ Download

Link: https://gitlab.com/multiloop-pku/amflow

Name	Last commit	Last update
diffeq_solver	update	5 months ago
examples	update	3 months ago
ibp_interface	fix_a_bug_for_mpi_version	1 week ago
AMFlow.m	fix mass mode	2 months ago
CHANGELOG.md	update changelog	1 week ago
FAQ.md	update	6 months ago
LICENSE.md	test	7 months ago
README.md	update	3 months ago
options_summary	update	3 months ago

➢ Description

- **The first (method and) package that can calculate any FI (with any number of loops, any D and $\vec{s}$) to arbitrary precision,** ***given sufficient resource***

Advantages: all purposes

➢ Expansion of D around any fixed value D_0

- Calculate FIs with $D = D_0 - 2\epsilon$ for a list of small ϵ (e.g. $0.01, 0.011, 0.012, \ldots, 0.02$)
- Fit Lauran expansion in ϵ
- D_0 can be 4, 3 (nonrelativistic theory), or other values
- Can obtain ϵ expansion to any order

X. Liu, YQM, 2201.11669
Z. F. Liu, YQM, 2201.11637

➢ Calculate FIs with linear propagators

- Present frequently in effective field theory

Z. F. Liu, YQM, 2201.11636

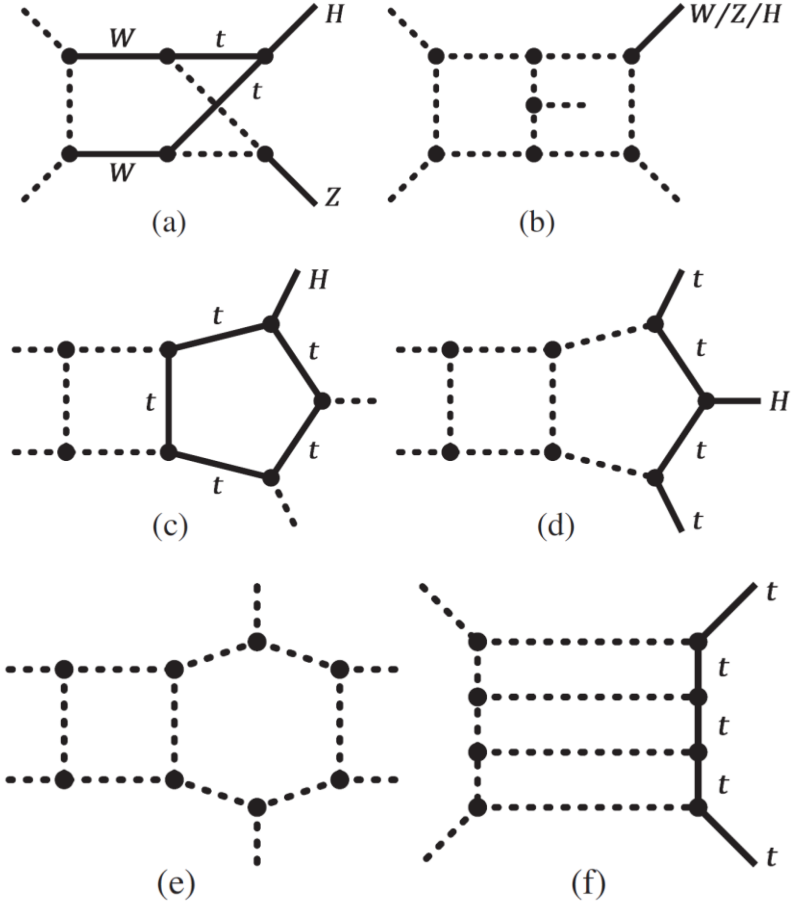
➢ Calculate phase space integrals

- As far as there is no jet

X. Liu, YQM, W. Tao, P. Zhang, 2009.07987

Examples using AMF

➢ Cutting-edge problems



X. Liu, YQM, 2107.01864

Family	dp	a	b	c	d	e	f
T_{setup}	6	20	18	8	1	25	30
T_{solve}	7	11	15	6	3	15	42
P_1	95%	99%	96%	99%	98%	94%	93%
$T_{\vec{s}}$	2	916	64	1305	30	1801	63

Time to setup DEs (CPU core hours)

- Results: 16-digit precision, to $\mathcal{O}(\epsilon^4)$
- First step of iteration: cost most time
- All results in (a)-(f) are new, very challenging for all other methods!
- Highly nontrivially checked!
- IBP reduction (bottleneck): C++
- Solve η-DEs: Mathematica. Can be significantly improved

Pheno. applications of AMF

➢ Two ways to use AMF

- **Use AMF to calculate each phase-space point**
- **Use AMF to generate BCs of $\vec{s}$-DEs**

➢ Wide range of applications

- **Linear propagators; Phase space integrals; QCD sum rules; Electroweak corrections; Quarkonia production; Complex mass; …**

AMFlow: A Mathematica package for Feynman integrals computation via auxiliary mass flow

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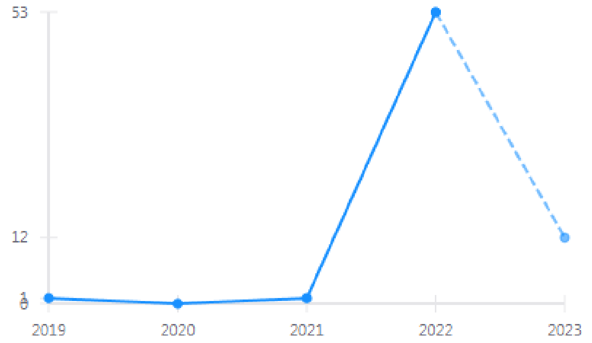

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Current status: master integrals calculation

➢ Main methods

- **Sector decomposition** — Hepp, (1966); Binoth, Heinrich, 0004013; F. Feng, Z. Li, …
- **Mellin-Barnes representation** — Usyukina (1975); Smirnov, 9905323; J. Wang, …
- **Difference equations** — Laporta, 0102033; Lee, 0911.0252
- **$\boxed{\text{Differential equations}}$** — Kotikov, PLB(1991)

Analytical : if ϵ-form exists

Henn, 1304.1806
L.B Chen, L.L. Yang, G. Yang, Y. Zhang, …

Numerical: general and efficient

X. Liu, YQM, C. Y. Wang, 1711.09572

➢ In future

1. **Find out the minimal number of MIs (D or 4): exhausting relations, simpler DEs**
2. **Define better MIs: simpler DEs, more efficient to get and solve numerically**
3. **Analytical: elliptic functions**

Current status: integral reduction

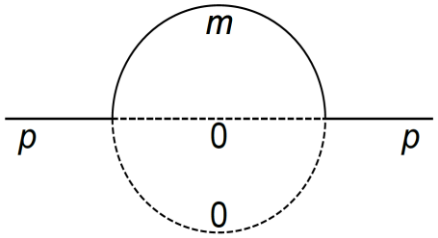
- **Difficulties of IBP method (although simply linear algebra)**
  - Complicated intermediate express
  - Very large scale of linear equations (can be billions of) E.g., Laporta 1910.01248
- **Improvements for IBPs**
  - Finite field: solving intermediate express swell
    Manteuffel, Schabinger, 1406.4513
  - Syzygy equations: trimming IBP system
    Gluza, Kajda, Kosower, 1009.0472
    Larsen, Zhang, et. al., 1511.01071, 1805.01873, 2104.06866
  - Block-triangular form: minimize IBP system (needs input)
    Liu, YQM, 1801.10523, Guan, Liu, YQM, 1912.09294
- **In future**
  1. A better way: combine syzygy equations and block-triangular form
  2. Find direct solution of IBP relations. IBP generators: a noncommutative algebra

Ways to bypass IBPs

- $1/D$ **expansion and matching**

Baikov, Chetyrkin, Kuhn, 0108197
Baikov, NPB (2003)
Baikov, 0507053

m=0: $I_{111} = -\left(\frac{4}{p^4}\right)^{-d/2}\left(1+\frac{13}{4d}+\frac{281}{32d^2}+\frac{2823}{128d^3}+\cdots\right)$



- $1/\eta$ **expansion and matching**

Guan, Liu, Ma, 1801.10523, 1912.09294
Wang, Li, Basat, 1901.09390, 2102.08225

$$I_{111}^{\rm aux} = \eta^{D-3}\left\{\left[\frac{(D-2)^2}{3D}\frac{p^2}{\eta}\right] I_{2,1}^{\rm bub} + \left[1+\frac{D-3}{3}\frac{m^2}{\eta} - \frac{(D+4)(D-3)}{9D}\frac{p^2}{\eta}\right] I_{2,2}^{\rm bub} + \mathcal{O}(\eta^{-2})\right.$$

- **Intersection theory**

Frellesvig, et. al., 1901.11510, 1907.02000
L.L. Yang,..

- **FIs** $I_{a_1,a_2,\ldots,a_N} \equiv K\int_{\mathcal{C}} u\,\varphi \equiv K\,\langle\varphi|\mathcal{C}]_\omega$

$$\varphi \equiv \hat\varphi\, d^N\mathbf{z}\,, \qquad \hat\varphi \equiv \frac{1}{z_1^{a_1} z_2^{a_2}\cdots z_N^{a_N}}\,, \qquad d^N\mathbf{z} \equiv dz_1\wedge dz_2\wedge\cdots\wedge dz_N$$

- **Intersection number** $\langle\varphi_L|\varphi_R\rangle_\omega = \sum_{p\in\mathcal{P}} {\rm Res}_{z=p}\left(\psi_p\,\varphi_R\right)$

Still hard,

worth further study

State-of-the-art perturbative computation

- 2→2 process with massive particles at two-loop: almost done

$$g + g \to t + \bar{t}, \qquad g + g \to H + H(g), \qquad e^+ e^- \to H\,Z$$

- Frontier in the following decade:
  - 2→3 processes at two loops (3j/γ, V/H+2j $t\bar{t}$+j, $t\bar{t}H$,…)
  - 2→2 processes at three loops (2j/γ, V/H+j, $t\bar{t}$, HH, …)
  - 2→1 processes at four loops (j, V/H, AP kernel)
  - Very a few obtained, usually no exact pure virtual contribution

- Very challenging

Davies, Herren, Steinhauser, 1911.10214

  - Four-loop $g + g \to H$ (NNLP in HTL): 860 days (wall time!)
  - Bottleneck: IBP reduction

Packages for fully automatic calculation

	Generate amplitudes	Manipulate amplitudes	Integral reduction	Master integrals calculation
Package used	FeynArts or qgraf	CalcLoop	Blade	AMFlow
Notes	https://feynarts.de/ http://cfif.ist.utl.pt/~paulo/qgraf.html	https://gitlab.com/yqma/CalcLoop	https://gitlab.com/multiloop-pku/blade	https://gitlab.com/multiloop-pku/amflow

Implementing block-triangular form, usually improves efficiency by $O(10^2)$

- Fully automatic, valid to any-loop order
- The key: AMFlow
- Main challenge: integral reduction is time/resource consuming

The dawn of automatic multi-loop calculation!

Automatic NLO correction obtained more than 10 years ago: MadGraph, Helac, FDC, etc

Summary for DimReg

- Feynman integrals form a finite-dim. linear space

- AMF: Feynman integrals can be completely determined once relations in the linear space is clear

- Result in a powerful method to calculate FIs: in principle any FI can be systematically calculated

  Impossible $\xRightarrow{\text{2022}}$ possible $\xRightarrow{\text{future}}$ efficiency

- Solving the linear space is still hard, stay tuned

Outlook

Integrals in lattice regulator

Nonperturbative computation of lattice correlation functions by differential equations

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We show that methods developed in the context of perturbative calculations can be transferred to nonperturbative calculations. We demonstrate that correlation functions on the lattice can be computed with the method of differential equations, supplemented with techniques from twisted cohomology. We derive differential equations for the variation with the coupling or—more generally—with the parameters of the action. Already simple examples show that the differential equation with respect to the coupling has an essential singularity at zero coupling and a regular singularity at infinite coupling. The properties of the differential equation at zero coupling can be used to prove that the perturbative series is only an asymptotic series.

DOI: 10.1103/PhysRevD.107.014502

Integrals and differential equations

$$S_E = \sum_{x\in\Lambda}\left(-\sum_{\mu=0}^{D-1}\phi_x\phi_{x+ae_\mu} + D\phi_x^2 + \sum_{j=2}^{j_{\max}}\frac{\lambda_j}{j!}\phi_x^j\right)$$

We are interested in the lattice integrals

$$I_{\nu_1\nu_2\ldots\nu_N} = \int_{\mathcal{C}^N} d^N\phi\left(\prod_{k=1}^{N}\phi_{x_k}^{\nu_k}\right)\exp(-S_E). \qquad (2)$$

The integration contour $\mathcal{C}$ is a curve in $\mathbb{C}$ and the same for every field variable ϕ_x. The integration contour is chosen such that $\exp(-S_E)$ goes to zero as we approach the boundary. The correlation functions are then given by

- **Using twisted cohomology (or IBP)**

$$\frac{d}{d\lambda_j}I_i = \sum_{k=1}^{N_F} A_{ik} I_k.$$

- **Boundary condition at $\lambda \to 0$ can be provided by perturbative calculation**

Works

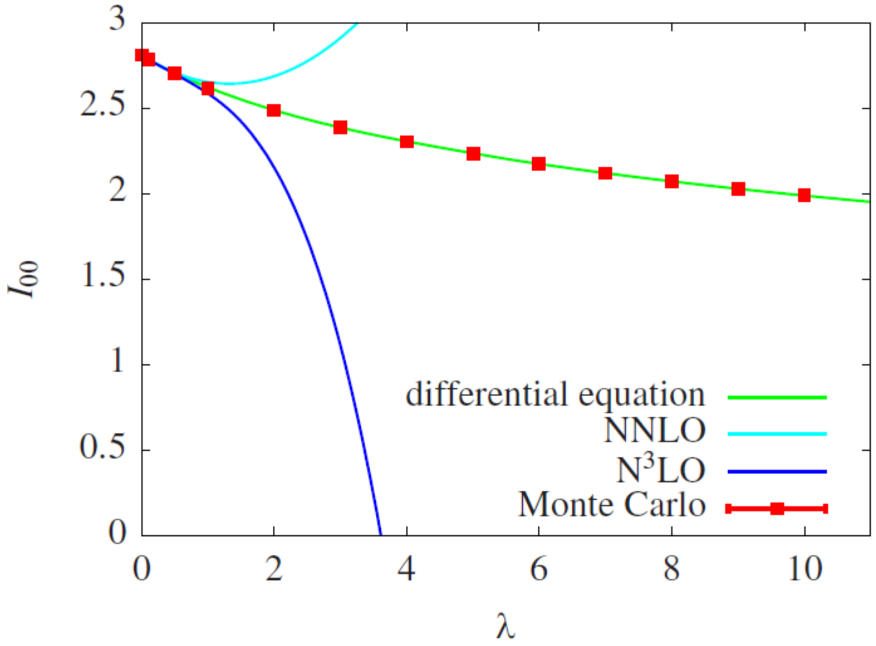
V. EXAMPLE 2: ϕ^4-THEORY

As our second example we consider massive ϕ^4-theory in $D = 1$ space-time dimensions with $L = 2$ lattice points. We set $\lambda_2 = m^2$ and $\lambda_4 = \lambda$. The action is given by

$$S_E = (\phi_{x_1} - \phi_{x_2})^2 + \frac{m^2}{2}(\phi_{x_1}^2 + \phi_{x_2}^2) + \frac{\lambda}{24}(\phi_{x_1}^4 + \phi_{x_2}^4).$$

$$J_1 = I_{00}, \quad J_2 = I_{11}, \quad J_3 = I_{22},$$
$$J_4 = I_{20} + I_{02}, \quad J_5 = I_{10} + I_{01}, \quad J_6 = I_{21} + I_{12},$$
$$J_7 = I_{10} - I_{01}, \quad J_8 = I_{21} - I_{12}, \quad J_9 = I_{20} - I_{02}.$$

$$\frac{d}{d\lambda}\begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2\lambda} & -\frac{1}{\lambda} & 0 & \frac{2+m^2}{4\lambda} \\ 0 & -\frac{3(2+m^2)^2}{\lambda^2} - \frac{1}{\lambda} & -\frac{1}{\lambda} & \frac{3(2+m^2)}{\lambda^2} \\ -\frac{72}{\lambda^3} & -\frac{72(6+4m^2+m^4)}{\lambda^3} & -\frac{3(2+m^2)^2}{\lambda^2} - \frac{3}{2\lambda} & \frac{3(2+m^2)(72+\lambda)}{2\lambda^3} \\ \frac{3(2+m^2)}{\lambda^2} & \frac{18(2+m^2)}{\lambda^2} & \frac{2+m^2}{2\lambda} & -\frac{3(12+4m^2+m^4)}{2\lambda^2} - \frac{1}{\lambda} \end{pmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \end{pmatrix}$$



Very challenge for practical use

Also for ϕ^4-theory we give some indications for the required CPU time on a standard laptop for larger lattices: For example, it takes about 280 s to compute the differential equation for $D = 1$ space-time dimensions with $L = 8$ lattice points and about 1400 s to compute the differential equation for $D = 3$ space-time dimensions with $L = 2$ lattice points in each direction.

- Only 8 lattice points
- At least hundred thousand points needed for practice
- Note: cutting-edge perturbative calculations have 11 variables; 9-variable problems have been solved 20 years ago

Any idea to improve?

Thank you!