Effective theory meets lattice: the two- and three-particle scattering

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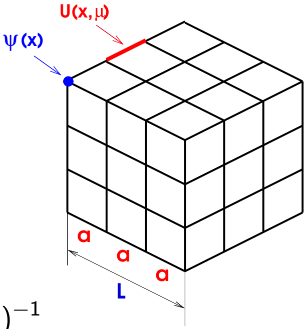
Plan

- Introduction
- Scattering observables: the no-go theorem
- Power counting and the choice of the EFT
- Two-body sector: Lüscher equation, the LL formalism
- Dimers and the three-body quantization condition
- Three-body decays
- Conclusions, outlook

QCD on the lattice

- In QCD, the structure of hadrons and their interactions at low energies cannot be studied in perturbation theory $\rightarrow$ QCD on the lattice

$$S = \frac{1}{g^2}\sum_{x\mu\nu} \mathrm{Re\,tr}(1 - P_{\mu\nu}(x))$$
$$+ \sum_{x\mu} \bar{\psi}\left(\frac{1}{2}\gamma_\mu(\nabla_\mu + \nabla^*_\mu) - \frac{a}{2}\nabla_\mu\nabla^*_\mu\right)\psi + \sum_x \bar{\psi}\, m\, \psi$$



- The covariant derivative and the plaquette:

$$\nabla_\mu\psi(x) = \frac{1}{a}\left(U(x,\mu)\psi(x + a\hat{\mu}) - \psi(x)\right)$$
$$P_{\mu\nu}(x) = U(x,\mu)U(x + a\hat{\mu},\nu)U(x + a\hat{\nu},\mu)^{-1}U(x,\nu)^{-1}$$
$$\mathrm{tr}(P_{\mu\nu}(x)) = N_c - \frac{1}{2}a^4\mathrm{tr}(G_{\mu\nu}(x)G_{\mu\nu}(x)) + O(a^5)$$

Masses of stable particles

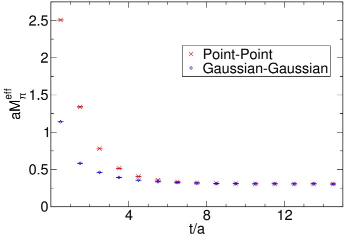
- Composite fields: $\phi_\pi(t,\mathbf{p}) = \sum_{\mathbf{x}} e^{-i\mathbf{p}\mathbf{x}} \bar{d}(x) i\gamma^5 u(x)$, $x = (\mathbf{x}, t)$
- The Euclidean path integral

$$D_\pi(t) = \langle T\phi_\pi(t,\mathbf{0})\phi_\pi^\dagger(0,\mathbf{0})|0\rangle = \frac{\int \mathscr{D}U\mathscr{D}\psi\mathscr{D}\bar{\psi} e^{-S}\, \phi_\pi(t,\mathbf{0})\phi_\pi^\dagger(0,\mathbf{0})}{\int \mathscr{D}U\mathscr{D}\psi\mathscr{D}\bar{\psi} e^{-S}}$$

- If $t \to \infty$, then

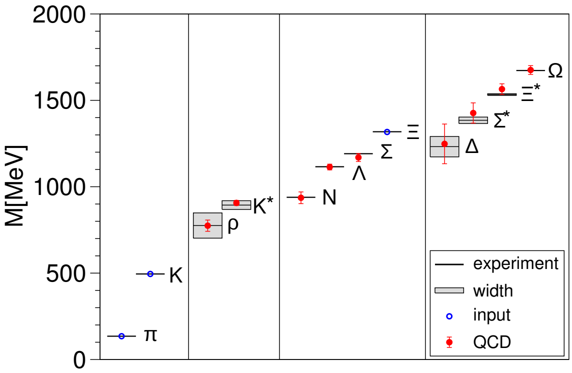
$$D_\pi(t) \to |\langle 0|\phi_\pi(0,\mathbf{0})|\pi\rangle|^2 e^{-M_\pi t} + \cdots$$

$$C_\pi(t) = \ln \frac{D_\pi(t)}{D_\pi(t+a)} \to aM_\pi + \cdots$$



S. Dürr *et al.*, Science 322 (2008) 1224

The low-energy spectrum of QCD



Meson and baryon spectrum in QCD, S. Dürr *et al.*, Science 322 (2008) 1224

The no-go theorem

- The scattering observables cannot be directly extracted from the amplitudes calculated on the lattice (Maiani and Testa, 1990)
- Example: the timelike form factor of the pion, $t', t \to \infty$ and $t' \gg t$:

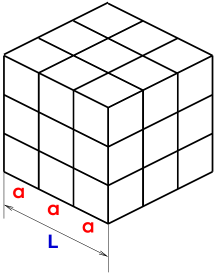
$$R_{\mathbf{p}-\mathbf{p}}(t', t) = \langle 0 | T \phi_\pi(t', \mathbf{p}) \phi_\pi(t, -\mathbf{p}) A_\mu(0) | 0 \rangle$$

$$\sim \sum_n e^{-w(\mathbf{p})t' - (E_n - w(\mathbf{p}))t} \langle 0 | \phi_\pi(0, \mathbf{p}) | \mathbf{p} \rangle \langle \mathbf{p} | \phi(0, -\mathbf{p}) | n \rangle \langle n | A_\mu(0) | 0 \rangle + \cdots$$

- The state with minimum energy: $E_n \to 2M_\pi < 2w(\mathbf{p}) = 2\sqrt{M_\pi^2 + \mathbf{p}^2}$
  $\hookrightarrow$ $\langle n | A_\mu(0) | 0 \rangle$ is not related to the form factor
- In a finite volume, the three-momentum is quantized
  $\hookrightarrow$ states lying above threshold can be reached

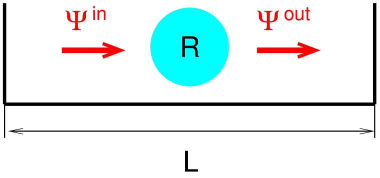
"Scattering" in a finite volume

- Impose (periodic) boundary conditions
- The spatial size of the box, L, is finite
- Assume the temporal size $L_t \gg L$, $L_t \to \infty$
- Three-momenta are quantized $\mathbf{p} = \dfrac{2\pi}{L}\mathbf{n}\,, \quad \mathbf{n} \in \mathbb{Z}^3$
- Discrete energy levels: $E_{n+1} - E_n = O(L^{-2})$
- No asymptotic states



How does one extract the scattering observables:
phase shift, cross section,. . . from the measured quantities?

EFT meets lattice



- When $R \ll L$, well-separated hadrons can be formed
- Scale separation is possible
- Since $p \sim 1/L$ and $R \sim 1/m$, then $p \ll m$: non-relativistic
  - Polarization effects, caused by creation/annihilation of the particles, are exponentially small and can be neglected
  - The relativistic kinematics should be implemented explicitly: non-rest frames

Non-relativistic EFT: essentials

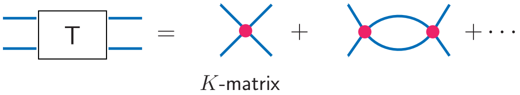
- Propagator:

$$\frac{1}{m^2 - p^2} = \underbrace{\frac{1}{2w(\mathbf{p})(w(\mathbf{p}) - p^0 - i\varepsilon)}}_{particle} + \underbrace{\frac{1}{2w(\mathbf{p})(w(\mathbf{p}) + p^0 - i\varepsilon)}}_{anti-particle}$$

- The vertices in the Lagrangian conserve particle number:

$$\mathscr{L} = \phi^\dagger(i\partial_t - w)(2w)\phi + \frac{C_0}{4}\,\phi^\dagger\phi^\dagger\phi\phi + \frac{D_0}{36}\,\phi^\dagger\phi^\dagger\phi^\dagger\phi\phi\phi + \cdots$$

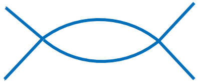
- Only bubble diagrams:



- Effective-range expansion: $K^{-1}(p) = p\cot\delta(p) = -\frac{1}{a} + \frac{1}{2}rp^2 + O(p^4)$

The Lüscher equation (Lüscher, 1991)

- Crucial point: $R \ll L$. The energy spectrum can be calculated by using the same EFT in a finite volume
- Couplings $C_0, D_0, \ldots$ describe the short-range physics. They are the same, according to the decoupling theorem
- Loop diagram in a finite volume



$$J_L(P) = \frac{1}{L^3}\sum_{\mathbf{k}}\int\frac{dk^0}{2\pi i}\,\frac{1}{2w(\mathbf{k})(w(\mathbf{k})-k^0-i\varepsilon)2w(\mathbf{P}-\mathbf{k})(w(\mathbf{P}-\mathbf{k})-P^0+k^0-i\varepsilon)}$$

Phase shift from the energy levels

- The Lüscher zeta-function:

$$J_L(P) \propto \frac{2}{\sqrt{\pi} L\gamma} Z_{00}^{\mathbf{P}}(1; q_0^2)\,, \qquad q_0 = \frac{pL}{2\pi}\,, \qquad q_0^2 = \frac{P^2}{4} - m^2$$

- The finite-volume spectrum is determined by the poles of the scattering T-matrix (the determinant of the linear equation vanishes)
- The Lüscher equation in the absence of partial-wave mixing:

$$p \cot\delta(p) = \frac{2}{\sqrt{\pi} L\gamma} Z_{00}^{\mathbf{P}}(1; q_0^2)$$

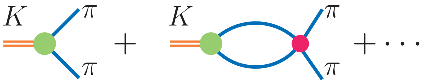
$\hookrightarrow$ measuring energy levels, one extracts phase shift at the same energy

- Resonances: analytic continuation into the complex plane

NREFT serves as a bridge between finite and infinite volume

The Lellouch-Lüscher formula (Lellouch & Lüscher, 2001)

- Final-state interactions lead to an irregular L-dependence of the matrix element



- The non-relativistic Lagrangian

$$\begin{aligned} \mathscr{L} &= \phi^\dagger(i\partial_t - w)(2w)\phi + \frac{C_0}{4}\phi^\dagger\phi^\dagger\phi\phi + \cdots + K^\dagger(i\partial_t - w_K)(2w_K)K \\ &+ g(K^\dagger\phi\phi + \text{h.c.}) \end{aligned}$$

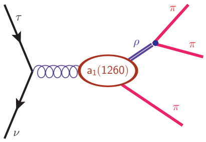
- Calculate the decay matrix element in a finite and in the infinite volume, extract g
- Matrix elements are related through

$$\langle n|H_W|K\rangle_L = \underbrace{\Phi_2(L)}_{\textit{depends on phase shift}} \langle \pi\pi; out|H_W|K\rangle_\infty$$

From two to three particles

Why three particles on the lattice?

- Three-pion decays of K, η, ω
- $a_1(1260) \to \rho\pi \to 3\pi$ and $a_1(1420) \to f_0(980)\pi \to 3\pi$
- Decays of exotica: $X(3872)$, $Y(4260)$, ...
- Roper resonance: πN and $\pi\pi N$ final states
- Nuclear reactions
- Few-body physics



Lattice vs. continuum: observables

Infinite volume:

- Three-particle bound states
- Elastic scattering
- Rearrangement reactions, breakup
- The mass and width of the three-particle resonances
- Resonance matrix elements (complex): e.g., $\langle \pi\pi\pi | H_W | K \rangle$

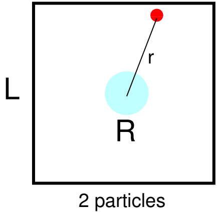
Finite volume:

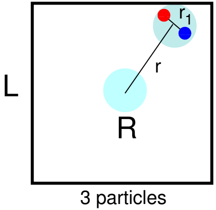
- Two- and three-particle energy levels
- Matrix elements between eigenstates (real)

How does one connect these two sets? EFT serves as a bridge!

Scale separation in the three-particle sector

Two-particle scattering: The wave function always in the asymptotic form near the walls: no off-shell effects!





- The three-particle wave function near the box walls is not always described by the asymptotic wave function
- Is the three-particle spectrum determined solely in terms of the S-matrix?

K. Polejaeva and AR, 2012: Yes!

Three-particle quantization condition

- Three different but equivalent formulations of the three-particle quantization condition are available
  - RFT (Relativistic Field Theory): Hansen & Sharpe, 2014
  - NREFT (Non-Relativistic Effective Field Theory): Hammer, Pang & AR, 2017
  - FVU (Finite-Volume Unitarity): Mai & Döring, 2017
- Enables one to extract scattering observables in the three-body sector from the measured finite-volume spectrum
- Alternative approaches: Briceño & Davoudi, 2013; Aoki *et al*, 2014; Guo, 2017

Particle-dimer picture

- Dimer: an alternative description of an infinite bubble sum;
- Mathematically equivalent to the standard treatment – not an approximation

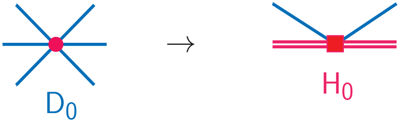
dimer : 

- Particle-dimer Lagrangian:

$$\mathscr{L} = \phi^\dagger(i\partial_t - w)(2w)\phi + \sigma T^\dagger T + \left(T^\dagger \left[\frac{f_0}{2}\phi\phi + \cdots \right] + \text{h.c.} \right)$$

- Higher partial waves can be included (introducing dimers with spin); Derivative terms should be added
- Matching: $f_0, \ldots \leftrightarrow C_0, \ldots \leftrightarrow a, r, \ldots$

Particle-dimer picture in the three-particle sector

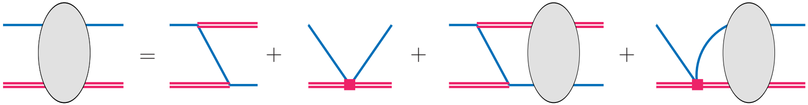


- The particle-dimer Lagrangian in the three-particle sector

$$\mathscr{L}_3 = h_0 T^\dagger T \phi^\dagger \phi + \cdots$$

- Matching: $h_0, \ldots \leftrightarrow D_0, \ldots$
- Terms with higher derivatives, higher dimer spin and orbital momentum should be added

The scattering equation in the infinite volume (CM frame)



Bethe-Salpeter equation

$$\mathcal{M}(\mathbf{p},\mathbf{q};E) = Z(\mathbf{p},\mathbf{q};E) + 8\pi \int^{\Lambda} \frac{d^3\mathbf{k}}{(2\pi)^3 2w(\mathbf{k})} Z(\mathbf{p},\mathbf{k};E)\tau(\mathbf{k};E)\mathcal{M}(\mathbf{k},\mathbf{q};E)$$

$$Z(\mathbf{p},\mathbf{q};E) = \frac{1}{2w(\mathbf{p}+\mathbf{q})(w(\mathbf{p})+w(\mathbf{q})+w(\mathbf{p}+\mathbf{q})-E)} + \tilde{H}_0 + \cdots$$

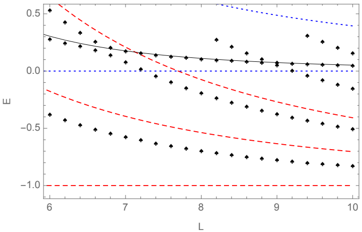
2-body amplitude: $4w(k^*)\tau^{-1}(\mathbf{k};E) = k^* \cot\delta(k^*) + \underbrace{\sqrt{\frac{s_2}{4} - m^2}}_{=k^*}$

Finite volume (CM frame)

$$\mathcal{M}_L(\mathbf{p},\mathbf{q};E) \;=\; Z(\mathbf{p},\mathbf{q};E) + \frac{8\pi}{L^3}\sum_{\mathbf{k}}^{\Lambda} Z(\mathbf{p},\mathbf{q};E)\frac{\tau_L(\mathbf{k};E)}{2w(\mathbf{k})}\mathcal{M}_L(\mathbf{k},\mathbf{q};E)$$

$$4w(k^*)\tau_L^{-1}(\mathbf{k};E) \;=\; k^*\cot\delta(k^*) - \frac{2}{\sqrt{\pi}L\gamma}Z_{00}^{\mathbf{P}}(1;q_0^2)\,, \qquad q_0 = \frac{k^*L}{2\pi}$$

- Poles in the amplitude $\rightarrow$ finite-volume spectrum: $\det((8\pi\tau_L)^{-1} - Z) = 0$



M. Döring *et al.*, 2018

Quantization condition: essentials

- Two-body interactions as an input: $k^* \cot\delta(k^*)$ fitted in the two-particle sector
- Extracting short-range quantities encoded in the three-body couplings $\tilde{H}_0, \ldots$ – should be fitted to the three-particle energies
- Finally, solve the equations in the infinite volume to arrive at the S-matrix elements!
- The approach is inherently three-dimensional: on-shell S-matrix elements are extracted
- Cubic box breaks relativistic invariance in a finite volume

What are the implications of the relativistic invariance?

Rewriting quantization condition in the invariant form

$$Z \to \frac{1}{2w(\mathbf{K}-\mathbf{p}-\mathbf{q})(w(\mathbf{p})+w(\mathbf{q})+w(\mathbf{K}-\mathbf{p}-\mathbf{q})-K^0)} + \underbrace{\frac{1}{2w(\mathbf{K}-\mathbf{p}-\mathbf{q})(w(\mathbf{p})+w(\mathbf{q})-w(\mathbf{K}-\mathbf{p}-\mathbf{q})-K^0)}}_{low-energy\ \ polynomial} + \tilde{H}_0 + \cdots$$

$$= \frac{1}{(p+q-K)^2-m^2} + \tilde{H}_0 + \cdots$$

- Conjecture: low-energy polynomial can be removed by renormalization
- The kernel is singular at high momenta, breaks unitarity already at threshold, gives rise to the spurious levels in a finite volume

Can the explicit Lorentz-invariance be reconciled with unitarity in the QC?

Explicitly Lorentz-invariant QC

(F. Müller, J.-Y. Pang, AR and J.-J. Wu, JHEP 02 (2022) 158)

- Choose "quantization axis" in direction of an arbitrary unit vector v^μ, $v^2 = 1$
- The Lagrangian:

$$\mathscr{L} = \phi^\dagger(i(v\partial) - w_v)(2w_v)\phi + \sum_\ell \sigma_\ell T^\dagger_{\mu_1\cdots\mu_\ell} T^{\mu_1\cdots\mu_\ell} + \frac{1}{2}\left(\sum_\ell T^\dagger_{\mu_1\cdots\mu_\ell} O^{\mu_1\cdots\mu_\ell} + \text{h.c.}\right)$$

- Here, $w_v = \sqrt{m^2 + \partial^2 - (v\partial)^2}$ and $O^{\mu_1\cdots\mu_\ell}$ denote the covariant operators, constructed out of two ϕ fields
- The propagator:

$$\langle 0|T\phi(x)\phi^\dagger(x)|0\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{2w_v(k)(w_v(k) - (vk) - i\varepsilon)}$$

The threshold expansion and the loop

There is no explicit v^μ-dependence of the loop

$$I(P) = \int \frac{d^D k}{(2\pi)^D i} \frac{1}{2w_v(k)(w_v(k) - (vk) - i\varepsilon)2w_v(P-k)(w_v(P-k) - (vP - vk) - i\varepsilon)}$$

$$\frac{1}{2w_v(k)(w_v(k) - (vk) - i\varepsilon)} = \frac{1}{m^2 - k^2} + \underbrace{\frac{1}{2w_v(k)(w_v(k) + (vk) - i\varepsilon)}}_{low-energy\ \ polynomial} + \cdots$$

Hence, $\quad I(P) = \int \frac{d^D k}{(2\pi)^D i} \frac{1}{(m^2 - k^2)(m^2 - (P-k)^2 - i\varepsilon)} + \cdots = \frac{\sigma}{16\pi^2} \ln \frac{\sigma - 1}{\sigma + 1} + \cdots$

$$\sigma = \left(1 - \frac{4m^2}{P^2 + i\varepsilon}\right)^{1/2}$$

Relativistic invariant QC in the three-body sector

$$\mathcal{M}_L(p,q) = Z(p,q) + \frac{8\pi}{L^3}\sum_{\mathbf{k}} \theta(\Lambda^2 + m^2 - (vk)^2) Z(p,k) \frac{\tau_L(P-k)}{2w(\mathbf{k})} \mathcal{M}_L(k,q)$$

$$\tau_L(P) = \frac{2\sqrt{P^2}}{p^* \cot\delta(p^*) - \frac{2}{\sqrt{\pi}L\gamma} Z_{00}^{\mathbf{P}}(1;q_0^2)}$$

$$Z(p,q) = \frac{1}{2w_v(K-p-q)(w_v(p)+w_v(q)+w_v(K-p-q)-(vK)-i\varepsilon} + \tilde{H}_0 + \cdots$$

- Quantization condition:

$$\det \mathscr{A} = 0\,, \qquad \mathscr{A}_{pq} = L^3 2w(\mathbf{p})\delta^3_{\mathbf{pq}}(8\pi\tau_L(K-p))^{-1} - Z(p,q)$$

- Relativistic invariance is achieved by choosing $v^\mu = K^\mu/\sqrt{K^2}$

Three-particle decays (F. Müller and AR, JHEP 03 (2021) 152)

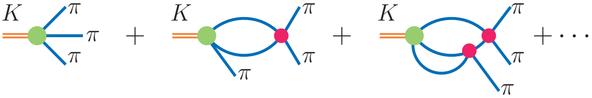
a) Decays through the weak or electromagnetic interactions; isospin-breaking decays: pole on the real axis

Example: $K \to 3\pi$

b) Decays through strong interactions, the pole moves into the complex plane

Example: $N(1440) \to \pi\pi N$

- Final-state interactions lead to the irregular volume-dependence in the matrix element



An analog of the LL formula in the three-particle sector?

The Lagrangian describing decays

- Three-particle picture:

$$\begin{aligned}\mathscr{L} &= \phi^\dagger(i\partial_t - w)(2w)\phi + K^\dagger(i\partial_t - w_K)(2w_K)K \\ &+ \frac{C_0}{4}\phi^\dagger\phi^\dagger\phi\phi + \frac{D_0}{36}\phi^\dagger\phi^\dagger\phi^\dagger\phi\phi\phi + \frac{G_0}{6}(K^\dagger\phi\phi\phi + \text{h.c.}) + \cdots\end{aligned}$$

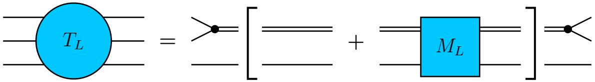
- Particle-dimer picture

$$\begin{aligned}\mathscr{L} &= \phi^\dagger(i\partial_t - w)(2w)\phi + K^\dagger(i\partial_t - w_K)(2w_K)K \\ &+ \frac{f_0}{2}(T^\dagger\phi\phi + \text{h.c.}) + h_0 T^\dagger T\phi^\dagger\phi + g_0(K^\dagger T\phi + \text{h.c.}) + \cdots\end{aligned}$$

The wave function

$$\mathcal{M}_L(\mathbf{p},\mathbf{q};K^0)\Big|_{K^0\to E_n} = \frac{\psi_L^{(n)}(\mathbf{p})\psi_L^{(n)}(\mathbf{q})}{E_n-K^0} + \text{regular}$$

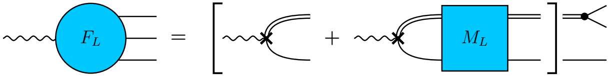
$$\psi_L^{(n)}(\mathbf{p}) = \frac{8\pi}{L^3}\sum_{\mathbf{k}}^{\Lambda} Z(\mathbf{p},\mathbf{k};E_n)\frac{\tau_L(\mathbf{k};E_n)}{2w(\mathbf{k})}\,\psi_L^{(n)}(\mathbf{p})$$



$$T_L(\{\mathbf{p}\},\{\mathbf{q}\};K^0)\Big|_{K^0\to E_n} = \frac{\Psi_L^{(n)}(\{\mathbf{p}\})\Psi_L^{(n)}(\{\mathbf{q}\})}{E_n-K^0} + \text{regular}$$

$$\Psi_L^{(n)}(\{\mathbf{p}\}) = \sum_{\alpha=1}^{3} 8\pi\tau_L(-\mathbf{p}_\alpha,E_n)\psi_L^{(n)}(-\mathbf{p}_\alpha)$$

Derivation of the three-particle LL formula



- Finite volume:

$$L^{3/2}|\langle n|H_W|K\rangle_L| = \left| \frac{g_0}{f_0} \frac{8\pi}{L^3} \sum_{\mathbf{q}} \psi_L^{(n)}(-\mathbf{q}) \frac{\tau_L(-\mathbf{q}; E_n)}{2w(\mathbf{q})} \right|$$

- Infinite volume:

$$\langle \pi(k_1)\pi(k_2)\pi(k_3); out|H_W|K\rangle_\infty$$

$$= \frac{g_0}{f_0} \sum_{\alpha=1}^{3} 8\pi\tau(-\mathbf{k}_\alpha; K^0)\left(1 + 8\pi \int^\Lambda \frac{d^3\mathbf{q}}{(2\pi)^3} \mathcal{M}(-\mathbf{k}_\alpha, -\mathbf{q}; K^0) \frac{\tau(-\mathbf{q}; K^0)}{2w(\mathbf{q})}\right)$$

The 3-particle LL factor

$$\langle \pi(k_1)\pi(k_2)\pi(k_3); out|H_W|K\rangle_\infty = \Phi_3(\{\mathbf{k}\})L^{3/2}\langle n|H_W|K\rangle_L$$

$$\Phi_3(\{\mathbf{k}\}) = \pm \frac{\sum_{\alpha=1}^{3} 8\pi\tau(-\mathbf{k}_\alpha; K^0)\left(1 + 8\pi \int^\Lambda \frac{d^3\mathbf{q}}{(2\pi)^3} \mathcal{M}(-\mathbf{k}_\alpha, -\mathbf{q}; K^0)\frac{\tau(-\mathbf{q}; K^0)}{2w(\mathbf{q})}\right)}{\frac{8\pi}{L^3}\sum_{\mathbf{q}} \psi_L^{(n)}(-\mathbf{q})\frac{\tau_L(-\mathbf{q}; E_n)}{2w(\mathbf{q})}}$$

- At lowest order, the coupling g_0 describes the short-range part of the $K \to 3\pi$ amplitude.
- The derivative couplings $g_1, g_2, \ldots$ emerge at higher orders. The three-particle LL factor becomes a matrix

Conclusions & outlook

- In the analysis of lattice data, EFT can be used to systematically relate the finite- and infinite-volume observables. This facilitates the extraction of scattering observables from lattice data
- The crucial point: decoupling of short- and long-range physics

Ex.1 Even the Lorentz invariance is explicitly broken in a finite volume, the extraction of the scattering observables can be performed in a manifestly invariant form

Ex.2 Using EFT, the power-law volume dependence of the three-particle decay amplitude can be explicitly calculated – a 3-particle analog of the LL formula

- Outlook
  - Long-range forces in a finite volume: one-pion exchange, Coulomb force
  - The Roper resonance
  - Boxed exotica

Thank you very much for your attention!

感谢各位聆听