

Hadronic contributions to the muon $g_\mu - 2$ in a dispersive approach

in collab. with M. Hoferichter and B. Kubis

JHEP **1908** (2019) 137 [arXiv:1907.01556 [hep-ph]]

and with S. Leupold and S. P. Schneider

JHEP **1810** (2018) 141, [arXiv:1808.04823 [hep-ph]]

Phys. Rev. Lett. **121** (2018) 112002, [arXiv:1805.01471 [hep-ph]]

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Outline

Muon anomaly & dispersion relations

Hadronic vacuum polarization

Overview & dispersive approach

3π contribution

Hadronic light-by-light scattering

Overview & dispersive approach

Pion-pole contribution

Conclusion and outlook

Muon anomaly & dispersion relations

Tests of the Standard Model:

Muon anomaly & dispersion relations

Tests of the Standard Model:

Energy frontier



{ Higgs boson
LHC, 2012
...

Muon anomaly & dispersion relations

Tests of the Standard Model:

Energy frontier



{ Higgs boson
LHC, 2012
...

Precision frontier



{ Muon $g_\mu - 2$
Fermilab, 2021
...

Muon anomaly & dispersion relations

The muon $g_\mu - 2$:

The magnetic dipole moment of a charged lepton ($\ell = e, \mu, \tau$):

$$\mu_\ell = g_\ell \left(\frac{Qe}{2m_\ell} \right) S$$

- Dirac equation universally predicts $g_\ell = 2$
- Quantum loop corrections lead to a deviation from the classical Dirac value 2

Define the lepton anomalous magnetic moment (lepton anomaly):

$$a_\ell \equiv \frac{g_\ell - 2}{2}$$

Muon anomaly & dispersion relations

The muon $g_\mu - 2$:

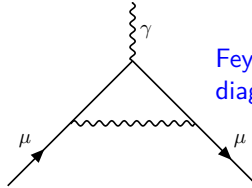
- A pure dimensionless number that can be both **theoretically** calculated and **experimentally** measured
- Contributions from a heavy scale M usually: $\propto m_\ell^2/M^2$
- τ is **so shortly** lived; a_μ is **more sensitive** than a_e by a factor of $\sim 4 \times 10^4$
- $a_\mu^{\text{SM}} = a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + \underbrace{a_\mu^{\text{Had}}}_{\substack{\bullet \text{ Hadronic vacuum polarization (HVP)} \\ \bullet \text{ Hadronic light-by-light (HLbL) scattering}}}$

Muon anomaly & dispersion relations

QED $\mathcal{O}(\alpha)$:



Schwinger, 1948



Feynman
diagrams

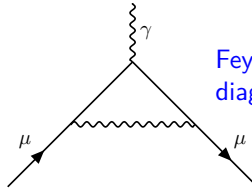
Feynman, 1948

Muon anomaly & dispersion relations

QED $\mathcal{O}(\alpha)$:



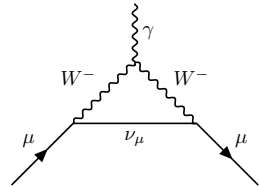
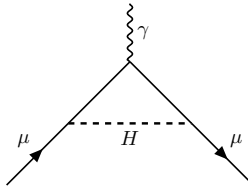
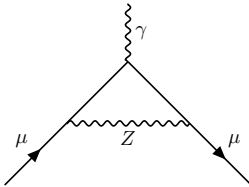
Schwinger, 1948



Feynman
diagrams

Feynman, 1948

Electroweak one-loop:



Muon anomaly & dispersion relations

Standard Model contributions to the muon $g_\mu - 2$:

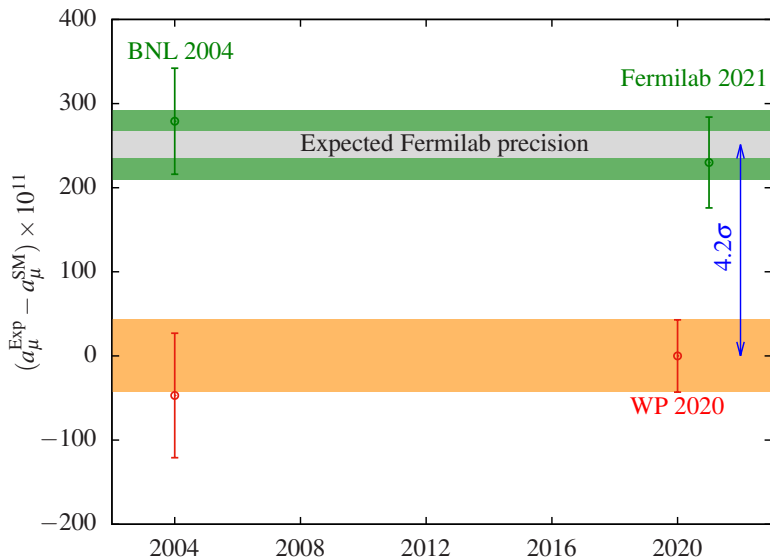
White paper result by the “Muon $g - 2$ Theory Initiative” (WP 2020)

Aoyama, ..., B.-L. H. et al., 2020

Contributions	$a_\mu \times 10^{11}$	$\Delta a_\mu \times 10^{11}$
QED (5 loops)	116584718.931	0.104
Electroweak (2 loops)	153.6	1.0
HVP (3 loops)	6845	40
HLbL (3 loops)	92	18
a_μ^{SM}	116591810	43
a_μ^{Exp}	116592061	41
$a_\mu^{\text{Exp}} - a_\mu^{\text{SM}}$	251	59

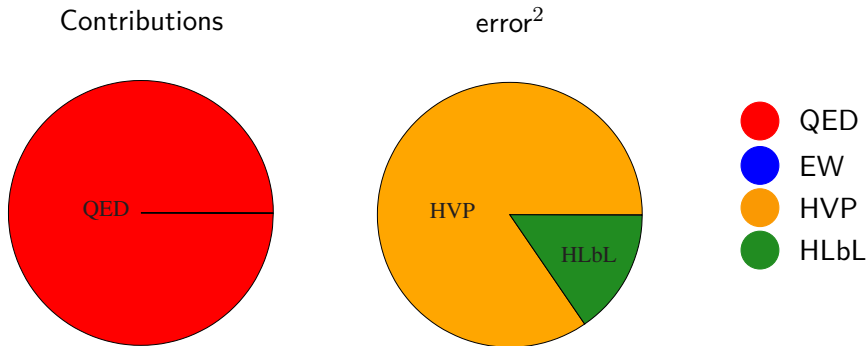
Muon anomaly & dispersion relations

$g_\mu - 2$ experiment vs theory:



Muon anomaly & dispersion relations

Standard Model theory:



Uncertainties mainly stem from **hadronic** contributions

Future experiments aim for four-fold improvement $\Delta a_\mu \approx 16 \times 10^{-11}$

⇒ With current theory $\sim 5.5\sigma$

Welcome **more precise** theoretical predictions

⇒ With theory improvements of same precision $\sim 11\sigma$!

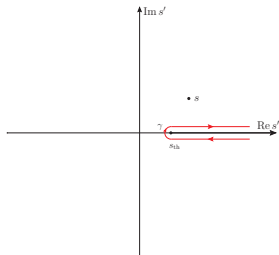
Muon anomaly & dispersion relations

Dispersion relations (DR):

- Microcausality $\Rightarrow$ analyticity
- Unitarity $\Rightarrow 2 \operatorname{Im} M_{fi} = \sum_n \int d\Pi_n M_{fn}^* M_{in}$
- Particle-antiparticle transformation $\Rightarrow$ crossing

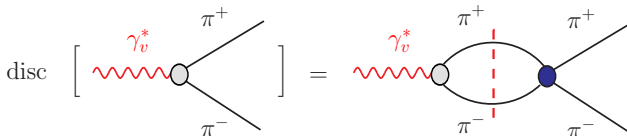
Combining analyticity, unitarity & crossing:

$$M(s) = \frac{1}{2\pi i} \int_{s_{\text{th}}}^{\infty} \frac{\text{disc } M(s')}{s' - s} ds' = \frac{1}{\pi} \int_{s_{\text{th}}}^{\infty} \frac{\operatorname{Im} M(s')}{s' - s} ds'$$



Muon anomaly & dispersion relations

Pion vector form factor $F_\pi^V(s)$:

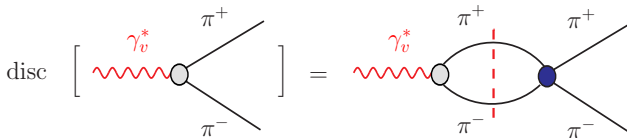


$$\text{disc } F_\pi^V(s) = 2i \text{Im } F_\pi^V(s) = 2i F_\pi^V(s) \sin \delta_1^1(s) e^{-i\delta_1^1(s)} \theta(s - 4M_\pi^2)$$

Watson's final-state theorem: phase of $F_\pi^V(s)$ is given by $\delta_1^1(s)$ Watson, 1954

Muon anomaly & dispersion relations

Pion vector form factor $F_\pi^V(s)$:



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Watson's final-state theorem: phase of $F_\pi^V(s)$ is given by $\delta_1^1(s)$ Watson, 1954

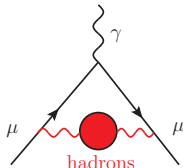
Solution:

$$F_\pi^V(s) = P(s) \Omega(s), \quad \Omega(s) = \exp \left\{ \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\delta_1^1(s')}{s'(s' - s)} \right\}$$

- $\Omega(s)$ is the Omnès function Omnès, 1958
- $P(s)$ polynomial, $P(0) = 1$ from charge conservation
- $\pi\pi$ P -wave phase shift $\delta_1^1(s)$ from Roy equations

Hadronic vacuum polarization

- m_μ as characteristic scale
⇒ **Not** a perturbative QCD problem!
- DR to relate to the observable



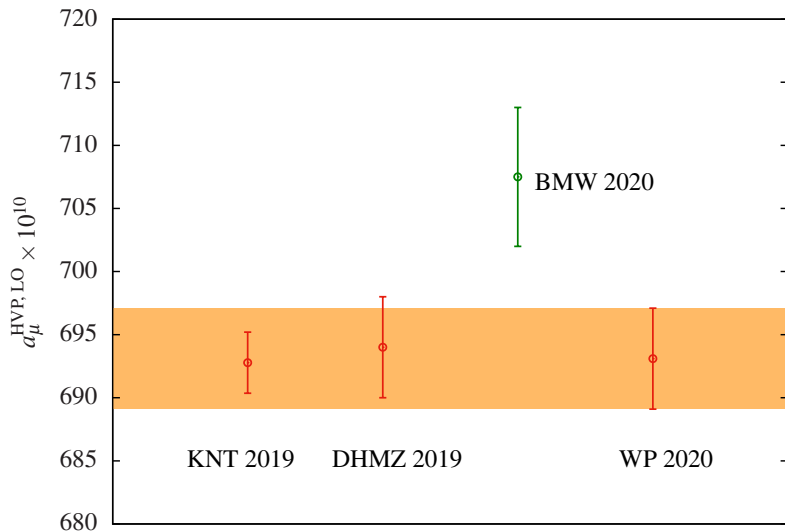
$$\text{Im} \text{ (hadron bubble) } \propto \left| \text{ (hadron bubble with outgoing lines) } \right|^2 \propto \sigma(e^+e^- \rightarrow \gamma^* \rightarrow \text{hadrons})$$

$$a_\mu^{\text{HVP}} = \int_{s_{\text{thr}}}^{\infty} ds K(s) \sigma^0(e^+e^- \rightarrow \text{hadrons})$$

- Kinematical function $K(s)$: $K(s) \propto 1/s$ for large s
- $\sigma^0(e^+e^- \rightarrow \text{hadrons}) \propto 1/s$ for large s

Hadronic vacuum polarization

Comparison between data-driven approaches and lattice:



Hadronic vacuum polarization

Complete data-driven evaluation needs:

- Exclusive channels $\pi^+\pi^-$, $\pi^+\pi^-\pi^0$, $\pi^0\gamma$, ...
- Inclusive data
- Perturbative QCD

Hadronic cross section:

- Inclusive of final state radiation
- Exempt from initial state radiation & vacuum polarization (VP)

Overview of the $\pi^+\pi^-\pi^0$ channel

- **Second largest** exclusive channel after $\pi^+\pi^-$
- **Large** relative discrepancy between direct data-integration works

$a_\mu^{3\pi} \times 10^{10}$ below 1.8 GeV before 2019		
Davier et al., 2017	Keshavarzi et al., 2018	relative difference
46.20 ± 1.45	47.70 ± 0.89	3.2% (0.7% for $\pi^+\pi^-$)

Even lower value $a_\mu^{3\pi}|_{\leq 2.0 \text{ GeV}} = 44.3(1.5) \times 10^{-10}!$ Jegerlehner, 2017

- Other independent analyses necessary

⇒ Dispersive global fit function fulfilling analyticity, unitarity & QCD constraints

$\gamma^* \rightarrow 3\pi$ dispersive representation

The $\gamma^*(q) \rightarrow \pi^+(p_+)\pi^-(p_-)\pi^0(p_0)$ decay amplitude $\mathcal{F}(s, t, u; q^2)$:

$$\langle 0 | j_\mu(0) | \pi^+(p_+)\pi^-(p_-)\pi^0(p_0) \rangle = -\epsilon_{\mu\nu\rho\sigma} p_+^\nu p_-^\rho p_0^\sigma \mathcal{F}(s, t, u; q^2)$$

$q = p_+ + p_- + p_0$; s, t & u are Mandelstam variables

Decompose into **single-variable** functions:

$$\mathcal{F}(s, t, u; q^2) = \mathcal{F}(s, q^2) + \mathcal{F}(t, q^2) + \mathcal{F}(u, q^2)$$

Normalization from the Wess–Zumino–Witten (WZW) anomaly:

$$\mathcal{F}(0, 0, 0; 0) = \frac{1}{4\pi^2 F_\pi^3} \equiv F_{3\pi}$$

$F_\pi = 92.28(10)$ MeV: pion decay constant

P. Zyla et al., 2020

$\gamma^* \rightarrow 3\pi$ dispersive representation

Discontinuity equation:

$$\text{disc } \mathcal{F}(s, q^2) = 2i(\mathcal{F}(s, q^2) + \hat{\mathcal{F}}(s, q^2))\theta(s - 4M_\pi^2) \sin \delta_1^1(s) e^{-i\delta_1^1(s)}$$

- $\mathcal{F}(s, q^2)$: right-hand cut
- $\hat{\mathcal{F}}(s, q^2)$: left-hand cut; angular averages of $\mathcal{F}(t, q^2)$ & $\mathcal{F}(u, q^2)$

$\gamma^* \rightarrow 3\pi$ dispersive representation

A **once-subtracted** dispersive solution to the discontinuity equation:

Hoferichter et al., 2014

$$\mathcal{F}(s, q^2) = a(q^2)\Omega(s)\left\{1 + \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\hat{\mathcal{F}}(s', q^2) \sin \delta_1^1(s')}{s'(s' - s)|\Omega(s')|}\right\}$$

$$\Omega(s) = \exp \left\{ \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\delta_1^1(s')}{s'(s' - s)} \right\}$$

is the Omnès function

Omnès, 1958

$\gamma^* \rightarrow 3\pi$ dispersive representation

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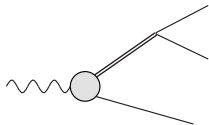
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Omnès, 1958

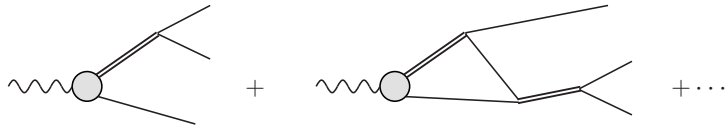
$\hat{\mathcal{F}}(s, q^2)$ absent:

$$\mathcal{F}(s, q^2) =$$



$\hat{\mathcal{F}}(s, q^2)$ present:

$$\mathcal{F}(s, q^2) =$$



- Incorporated **crossed-channel** interactions

$\gamma^* \rightarrow 3\pi$ dispersive representation

$a(q^2)$ fit to different $e^+e^- \rightarrow 3\pi$ cross-section data with parameterization:

$$a(q^2) = \frac{F_{3\pi}}{3} + \frac{q^2}{\pi} \int_{s_{\text{thr}}}^{\infty} ds' \frac{\text{Im } \mathcal{A}(s')}{s'(s' - q^2)} + C_n(q^2)$$

$$\mathcal{A}(q^2) = \sum_V \frac{{}^{cV}}{M_V^2 - q^2 - i\sqrt{q^2} \Gamma_V(q^2)}, \quad V = \omega, \phi, \omega', \omega''$$

$$C_n(q^2) = \sum_{i=1}^n c_i (z(q^2)^i - z(0)^i), \quad z(q^2) = \frac{\sqrt{s_{\text{inel}} - s_1} - \sqrt{s_{\text{inel}} - q^2}}{\sqrt{s_{\text{inel}} - s_1} + \sqrt{s_{\text{inel}} - q^2}}$$

$\gamma^* \rightarrow 3\pi$ dispersive representation

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- S -wave cusp eliminated
- **Exact** implementation of $\gamma^* \rightarrow 3\pi$ anomaly:

$$\frac{F_{3\pi}}{3} = \frac{1}{\pi} \int_{s_{\text{thr}}}^{\infty} ds' \frac{\text{Im } a(s')}{s'}$$

$e^+e^- \rightarrow 3\pi$ cross section data sets

Experiment	Region of $\sqrt{s}$ GeV	# data points	Normalization uncertainty
SND 2002	[0.98, 1.38]	67	5.0% or 5.4%
SND 2003	[0.66, 0.97]	49	3.4% or 4.5%
SND 2015	[1.05, 1.80]	31	3.7%
CMD-2 1995	[0.99, 1.03]	16	4.6%
CMD-2 1998	[0.99, 1.03]	13	2.3%
CMD-2 2004	[0.76, 0.81]	13	1.3%
CMD-2 2006	[0.98, 1.06]	54	2.5%
DM1 1980	[0.75, 1.10]	26	3.2%
ND 1991	[0.81, 1.39]	28	10% or 20%
DM2 1992	[1.34, 1.80]	10	8.7%
BaBar 2004	[1.06, 1.80]	30	all systematics

- Normalization-type systematic uncertainties are assumed to be 100% correlated
- Normalization uncertainties produce a **biased** fit for an empirical full covariance-matrix minimization

D'Agostini bias & unbiased fits

Simple example of overall normalization uncertainty inducing a bias:

D'Agostini, 1994

$y_1 = 8.0 \pm 2\%$ & $y_2 = 8.5 \pm 2\%$, normalization error of $\epsilon = 10\%$

- Covariance matrix

$$V = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} + \epsilon^2 \begin{pmatrix} y_1^2 & y_1 y_2 \\ y_1 y_2 & y_2^2 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 + \epsilon^2 y_1^2 & \epsilon^2 y_1 y_2 \\ \epsilon^2 y_1 y_2 & \sigma_2^2 + \epsilon^2 y_2^2 \end{pmatrix}$$

- $\chi^2 = \Delta^T V^{-1} \Delta$, $\Delta = \begin{pmatrix} y_1 - \hat{y} \\ y_2 - \hat{y} \end{pmatrix}$

- $\Rightarrow \hat{y} = 7.87 \pm 0.81 < y_1 \text{ \& } y_2$?

D'Agostini bias & unbiased fits

- Happens when the data values are rescaled independently of their errors
- Smaller data points are assigned a smaller uncertainty than larger ones

A better covariance matrix:

$$V = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} \hat{y}^2 & \hat{y}^2 \\ \hat{y}^2 & \hat{y}^2 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 + \varepsilon^2 \hat{y}^2 & \varepsilon^2 \hat{y}^2 \\ \varepsilon^2 \hat{y}^2 & \sigma_2^2 + \varepsilon^2 \hat{y}^2 \end{pmatrix}$$

General iterative solution:

NNPDF collaboration, 2010

$$V_{n+1}(i, j) = V^{\text{stat}}(i, j) + \frac{V^{\text{syst}}(i, j)}{y_i y_j} f_n(x_i) f_n(x_j)$$

Fit results

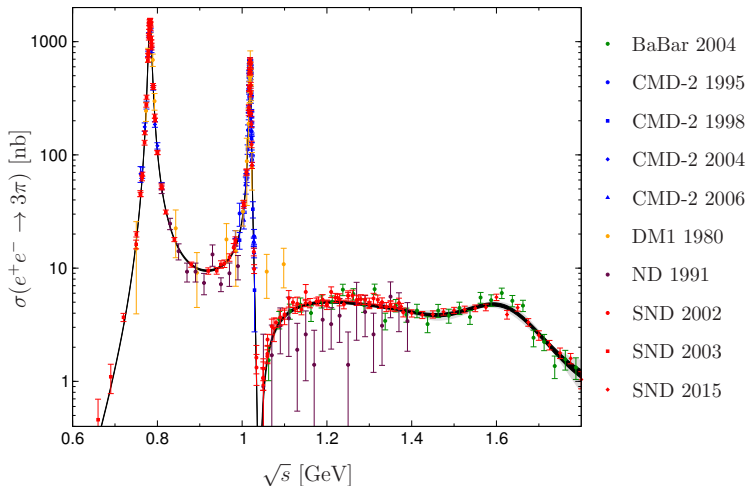
- Resonance parameters $M_\omega, \Gamma_\omega, M_\phi, \Gamma_\phi, c_\omega, c_\phi, c_{\omega'} \text{ \& } c_{\omega''}$
- Conformal parameters $c_1, c_2 \text{ \& } c_3$
- Energy rescaling $\sqrt{s} \rightarrow \sqrt{s} + \xi(\sqrt{s} - 3M_\pi)$

This work	
χ^2/dof	430.8/305=1.41

- Fit errors are inflated by the scale factor $S = \sqrt{\chi^2/\text{dof}}$

Fit results

Fit to $e^+e^- \rightarrow 3\pi$ data up to 1.8 GeV:



- VP removed from the cross section
- Black and Gray bands represent fit and total uncertainties

Fit results

Central result for the 3π contribution to HVP:

$$a_\mu^{3\pi}|_{\leq 1.8 \text{ GeV}} = 46.2(6)(6) \times 10^{-10} = 46.2(8) \times 10^{-10}$$

- **Interpolation errors** $\Rightarrow$ main discrepancy between different groups

Threshold region $a_\mu^{3\pi}|_{\leq 0.66 \text{ MeV}} = 0.02 \times 10^{-10}$

- **Twice** the estimate from WZW action+vector meson dominance model
Kuraev and Silagadze, 1995, Ahmedov et al., 2002

Fit results

Update after the current analysis:

$a_{\mu}^{3\pi} \times 10^{10}$ below 1.8 GeV in WP 2020		
Davier et al., 2019	Keshavarzi et al., 2019	Hoferichter et al., 2019
46.21 ± 1.45	46.63 ± 0.94	46.16 ± 0.82

- Large relative difference **resolved**

In combination with the 2π channel:

Colangelo et al., 2018

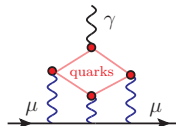
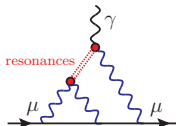
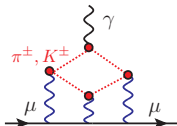
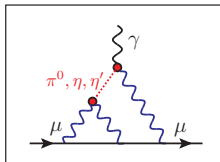
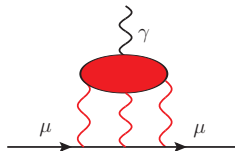
$$a_{\mu}^{2\pi}|_{\leq 1.0 \text{ GeV}} + a_{\mu}^{3\pi}|_{\leq 1.8 \text{ GeV}} = 541.2(2.7) \times 10^{-10}$$

- 80%** of HVP imposing analyticity and unitarity constraints

Hadronic light-by-light scattering

- Previous estimates based on **hadronic models** except for **lattice**

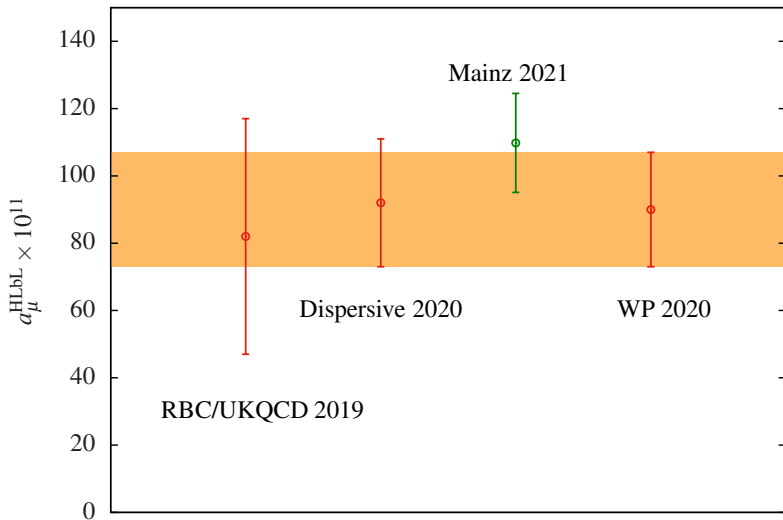
- New model-independent** initiatives: employ DR to relate dominant contributions to observables like **form factors**
Colangelo et al., 2014,
Pauk, Vanderhaeghen, 2014



- π^0 -pole term is the **largest individual** contribution to HLbL

Hadronic light-by-light scattering

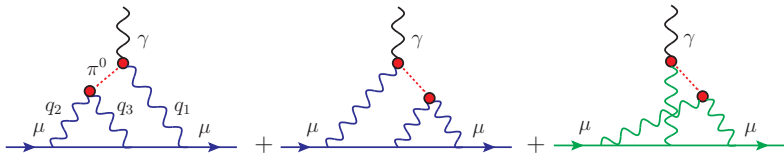
Comparison between dispersive approach and lattice:



Pion-pole contribution—definition

A 3 dimensional representation for the pion-pole contribution:

Knecht, Nyffeler, 2002
Jegerlehner, Nyffeler, 2009



$$a_{\mu}^{\pi^0\text{-pole}} = \left(\frac{\alpha}{\pi}\right)^3 \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau$$

$$\times \left[w_1(Q_1, Q_2, \tau) F_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -(Q_1 + Q_2)^2) F_{\pi^0\gamma^*\gamma^*}(-Q_2^2, 0) \right.$$

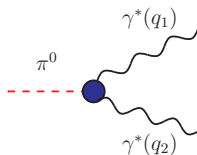
$$\left. + w_2(Q_1, Q_2, \tau) F_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) F_{\pi^0\gamma^*\gamma^*}(-(Q_1 + Q_2)^2, 0) \right]$$

- $w_1(Q_1, Q_2, \tau)$ & $w_2(Q_1, Q_2, \tau)$: weight functions; concentrated in $Q_i \leq 0.5 \text{ GeV}$
- $F_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$: on-shell space-like pion transition form factors

Pion transition form factor

- Defined by the matrix element of two electromagnetic currents $j_\mu(x)$

$$\begin{aligned} & i \int d^4x e^{iq_1 \cdot x} \langle 0 | T \{ j_\mu(x) j_\nu(0) \} | \pi^0(q_1 + q_2) \rangle \\ &= -\epsilon_{\mu\nu\rho\sigma} q_1^\rho q_2^\sigma F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) \end{aligned}$$



- Normalization fixed by the Adler–Bell–Jackiw anomaly:

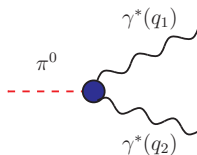
$$F_{\pi^0\gamma^*\gamma^*}(0,0) = \frac{1}{4\pi^2 F_\pi} \equiv F_{\pi\gamma\gamma}$$

Pion transition form factor

- Defined by the matrix element of two electromagnetic currents $j_\mu(x)$

$$i \int d^4x e^{iq_1 \cdot x} \langle 0 | T \{ j_\mu(x) j_\nu(0) \} | \pi^0(q_1 + q_2) \rangle$$

$$= -\epsilon_{\mu\nu\rho\sigma} q_1^\rho q_2^\sigma F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$$



- Normalization fixed by the Adler–Bell–Jackiw anomaly:

$$F_{\pi^0\gamma^*\gamma^*}(0, 0) = \frac{1}{4\pi^2 F_\pi} \equiv F_{\pi\gamma\gamma}$$

We build the form factor representation:

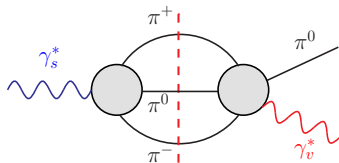
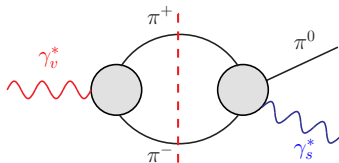
$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = F_{\pi^0\gamma^*\gamma^*}^{\text{disp}}(q_1^2, q_2^2) + F_{\pi^0\gamma^*\gamma^*}^{\text{eff}}(q_1^2, q_2^2) + F_{\pi^0\gamma^*\gamma^*}^{\text{asym}}(q_1^2, q_2^2)$$

- Reconstructed from all **lowest-lying** singularities
- Fulfills the asymptotic constraints at $\mathcal{O}(1/Q^2)$
- Suitable for a_μ loop-integral evaluation

Pion transition form factor

Dispersive reconstruction from the **lowest-lying** hadronic intermediate states:

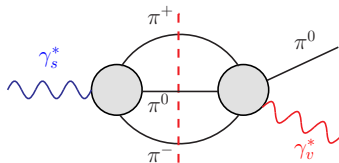
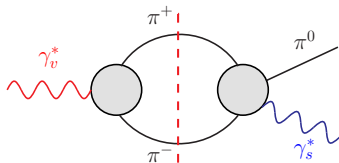
$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = F_{vs}(q_1^2, q_2^2) + F_{vs}(q_2^2, q_1^2)$$



Pion transition form factor

Dispersive reconstruction from the **lowest-lying** hadronic intermediate states:

$$F_{\pi^0 \gamma^* \gamma^*}(q_1^2, q_2^2) = F_{vs}(\textcolor{red}{q}_1^2, \textcolor{blue}{q}_2^2) + F_{vs}(\textcolor{red}{q}_2^2, \textcolor{blue}{q}_1^2)$$

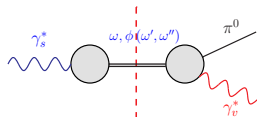


Isovector photon: 2 pions

- $\textcolor{red}{\gamma}_v^* \rightarrow \pi^+ \pi^- \rightarrow \textcolor{blue}{\gamma}_s^* \pi^0$
- $\text{disc} \propto \text{pion vector form factor}$
 $\times \textcolor{blue}{\gamma}_s^* \rightarrow 3\pi \text{ amplitude}$

Isoscalar photon: 3 pions

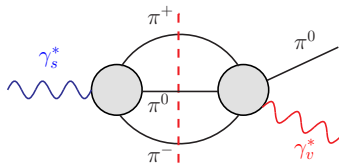
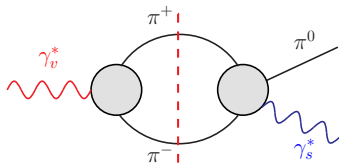
- $\textcolor{blue}{\gamma}_s^* \rightarrow \pi^+ \pi^- \pi^0 \rightarrow \textcolor{red}{\gamma}_v^* \pi^0$
- Dominated by resonances
 $\omega, \phi, \omega', \& \omega''$



Pion transition form factor

Dispersive reconstruction from the **lowest-lying** hadronic intermediate states:

$$F_{\pi^0 \gamma^* \gamma^*}(q_1^2, q_2^2) = F_{vs}(\textcolor{red}{q}_1^2, \textcolor{blue}{q}_2^2) + F_{vs}(\textcolor{red}{q}_2^2, \textcolor{blue}{q}_1^2)$$



Building blocks of the dispersive treatment:

- Pion vector form factor $F_\pi^V(s)$
- Partial wave amplitude $f_1(s, q^2)$ for the $\gamma_s^*(q) \rightarrow \pi^+ \pi^- \pi^0$ reaction

Pion transition form factor

Fit $a(q^2)$ to different $e^+e^- \rightarrow 3\pi$ cross-section data with:

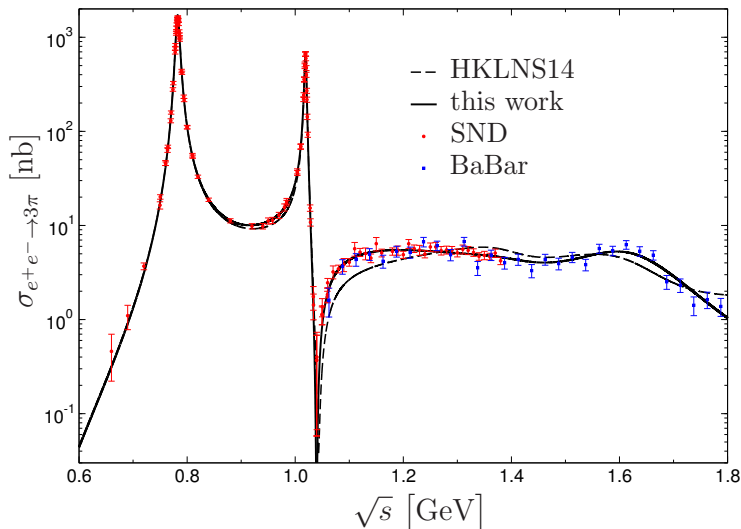
- S -wave cusp eliminated
- Exact implementation of $\gamma_s^*(q) \rightarrow 3\pi$ anomaly

$$\frac{F_{3\pi}}{3} = \frac{1}{\pi} \int_{s_{\text{thr}}}^{\infty} ds' \frac{\text{Im } a(s')}{s'}$$

- Asymptotic behavior of $C_n(q^2)$ controlled

Pion transition form factor

6 (7) parameters c_ω , c_ϕ , $c_{\omega'}$, $c_{\omega''}$, c_1 , c_2 & (c_3) fit to $e^+e^- \rightarrow 3\pi$ data:



- Substantially improved above the ϕ peak

Pion transition form factor

Double-spectral representation of the form factor:

$$F_{\pi^0 \gamma^* \gamma^*}^{\text{disp}}(-Q_1^2, -Q_2^2) = \frac{1}{\pi^2} \int_{4M_\pi^2}^{s_{\text{iv}}} dx \int_{s_{\text{thr}}}^{s_{\text{is}}} \frac{\rho^{\text{disp}}(x, y) dy}{(x + Q_1^2)(y + Q_2^2)},$$
$$\rho^{\text{disp}}(x, y) = \frac{q_\pi^3(x)}{12\pi\sqrt{x}} \text{Im} \left[(F_\pi^V(x))^* f_1(x, y) \right] + [x \leftrightarrow y]$$

$$f_1(s, q^2) = \mathcal{F}(s, q^2) + \hat{\mathcal{F}}(s, q^2): \gamma_s^*(q) \rightarrow 3\pi \text{ } P\text{-wave}$$

Pion transition form factor

Effective pole term:

$F_{\pi^0\gamma^*\gamma^*}^{\text{disp}}(q_1^2, q_2^2)$ fulfills the chiral anomaly $F_{\pi\gamma\gamma}$ by around 90%

⇒ Introduce an effective pole term

$$F_{\pi^0\gamma^*\gamma^*}^{\text{eff}}(q_1^2, q_2^2) = \frac{g_{\text{eff}}}{4\pi^2 F_\pi} \frac{M_{\text{eff}}^4}{(M_{\text{eff}}^2 - q_1^2)(M_{\text{eff}}^2 - q_2^2)}$$

g_{eff} fixed by fulfilling the chiral anomaly

$g_{\text{eff}} \sim 10\%$, ⇒ small

M_{eff} fit to singly-virtual data excluding BaBar above 5 GeV² Gronberg et al., 1998,
Aubert et al., 2009, Uehara et al., 2012

$M_{\text{eff}} \sim 1.5\text{--}2\text{ GeV}$, ⇒ reasonable

Pion transition form factor

Asymptotically, $F_{\pi^0\gamma^*\gamma^*}$ should fulfill

Brodsky, Lepage, 1979-1981

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = -\frac{2F_\pi}{3} \int_0^1 dx \frac{\phi_\pi(x)}{xq_1^2 + (1-x)q_2^2} + \mathcal{O}\left(\frac{1}{q_i^4}\right),$$

Pion distribution amplitude $\phi_\pi(x) = 6x(1-x) + \dots$

Brodsky–Lepage (BL) limit:

$$\lim_{Q^2 \rightarrow \infty} F_{\pi^0\gamma^*\gamma}(-Q^2, 0) = \frac{2F_\pi}{Q^2}$$

Operator product expansion (OPE):

Nesterenko, Radyushkin, 1983,
Novikov et al., 1984, Manohar, 1990

$$\lim_{Q^2 \rightarrow \infty} F_{\pi^0\gamma^*\gamma^*}(-Q^2, -Q^2) = \frac{2F_\pi}{3Q^2}$$

Pion transition form factor

Rewrite the asymptotic form into a **double-spectral** representation:

$$F_{\pi^0\gamma^*\gamma^*}^{\text{asym}}(q_1^2, q_2^2) = \frac{1}{\pi^2} \int_0^\infty \int_0^\infty dx dy \frac{\rho^{\text{asym}}(x, y)}{(x - q_1^2)(y - q_2^2)},$$

$$\rho^{\text{asym}}(x, y) = -2\pi^2 F_\pi xy \delta''(x - y)$$

Decomposition of the pion transition form factor:

$$\begin{aligned} F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) &= \frac{1}{\pi^2} \int_0^{s_m} dx \int_0^{s_m} dy \frac{\rho^{\text{disp}}(x, y)}{(x - q_1^2)(y - q_2^2)} + \frac{1}{\pi^2} \int_{s_m}^\infty dx \int_{s_m}^\infty dy \frac{\rho^{\text{asym}}(x, y)}{(x - q_1^2)(y - q_2^2)} \\ &+ \frac{1}{\pi^2} \int_0^{s_m} dx \int_{s_m}^\infty dy \frac{\rho(x, y)}{(x - q_1^2)(y - q_2^2)} + \frac{1}{\pi^2} \int_{s_m}^\infty dx \int_0^{s_m} dy \frac{\rho(x, y)}{(x - q_1^2)(y - q_2^2)} \end{aligned}$$

- s_m : continuum threshold
- $\rho(x, y)$ not known rigorously, $\rho^{\text{asym}}(x, y)$ applied in **mixed regions** vanishes
 $\Rightarrow$ All constraints can be fulfilled discarding **mixed regions**

Pion transition form factor

This defines the asymptotic contribution:

$$F_{\pi^0\gamma^*\gamma^*}^{\text{asym}}(q_1^2, q_2^2) = 2F_\pi \int_{s_m}^{\infty} dx \frac{q_1^2 q_2^2}{(x - q_1^2)^2 (x - q_2^2)^2}$$

- Does not contribute for the **singly-virtual** kinematics
- Restores the asymptotics for **singly-/doubly-virtual** kinematics

The final representation:

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = F_{\pi^0\gamma^*\gamma^*}^{\text{disp}}(q_1^2, q_2^2) + F_{\pi^0\gamma^*\gamma^*}^{\text{eff}}(q_1^2, q_2^2) + F_{\pi^0\gamma^*\gamma^*}^{\text{asym}}(q_1^2, q_2^2)$$

Numerical results

Uncertainty estimates:

- The uncertainty in $F_{\pi\gamma\gamma}$ at 0.8% from PrimEx II Larin et al., 2020
 - ▶ Varying the coupling g_{eff}
- Dispersive uncertainties estimated by
 - ▶ Varying the cutoffs between 1.8 and 2.5 GeV
 - ▶ Different $\pi\pi$ phase shifts Caprini et al., 2012, García-Martín et al., 2011, Schneider et al., 2012
 - ▶ Different representations of $F_{\pi}^V(s)$
 - ▶ Different conformal polynomial fits to $e^+e^- \rightarrow 3\pi$

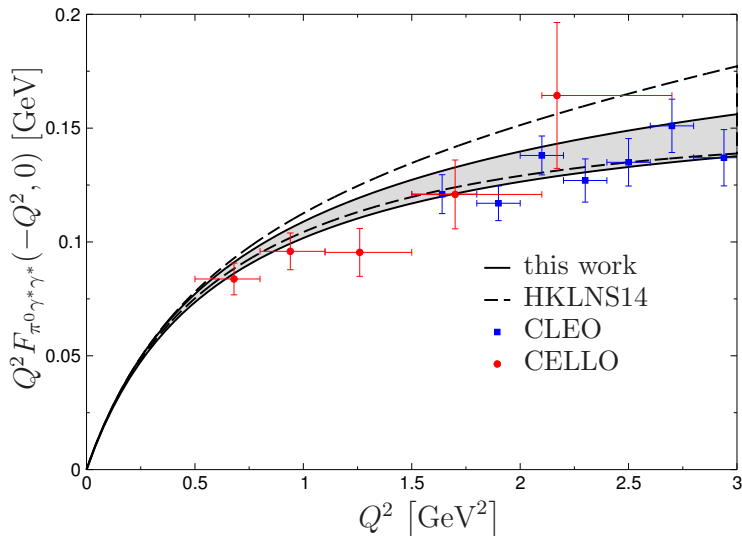
Numerical results

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 - ▶ Different representations of $F_{\pi}^V(s)$
 - ▶ Different conformal polynomial fits to $e^+e^- \rightarrow 3\pi$
- BL limit uncertainty by ${}^{+20}_{-10}\%$ Aubert et al., 2009, Uehara et al., 2012
 - ▶ Varying the mass parameter M_{eff}
 - ▶ Completely covers 3σ band
- Asymptotic part $s_m = 1.7(3) \text{ GeV}^2$ Khodjamirian, 1999, Agaev et al., 2011, Mikhailov et al., 2016
 - ▶ Expected from light-cone sum rules

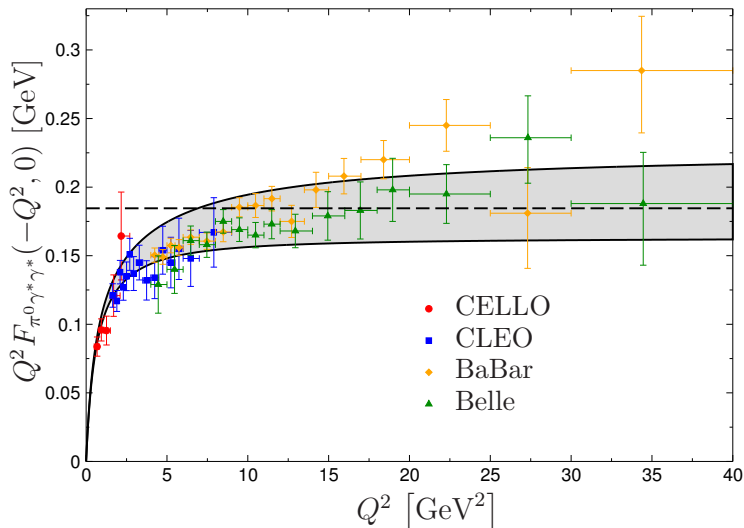
Numerical results

Singly-virtual **space-like** transition form factor in $Q^2 F_{\pi^0 \gamma^* \gamma^*}(-Q^2, 0)$:



Numerical results

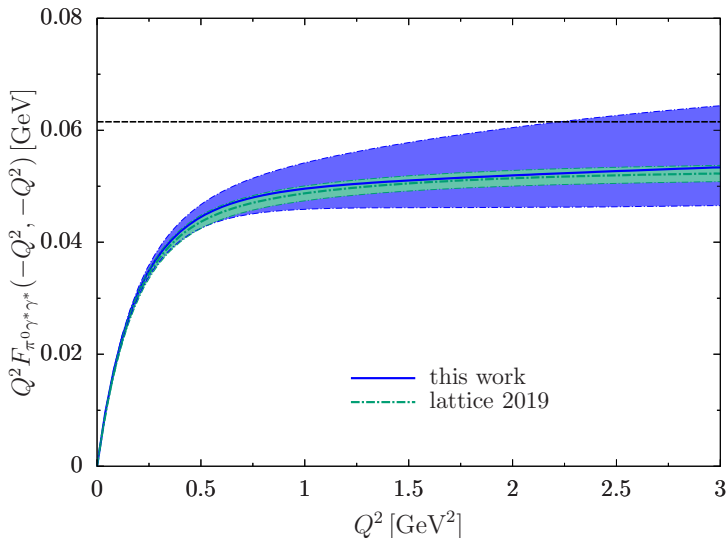
Singly-virtual **space-like** transition form factor in $Q^2 F_{\pi^0 \gamma^* \gamma^*}(-Q^2, 0)$:



Numerical results

Diagonal form factor in $Q^2 F_{\pi^0 \gamma^* \gamma^*}(-Q^2, -Q^2)$ in comparison to lattice:

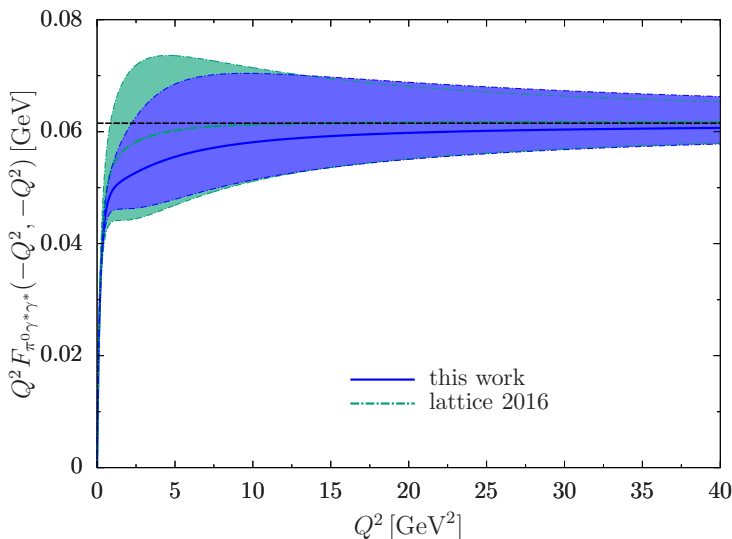
Gérardin et al., 2019



Numerical results

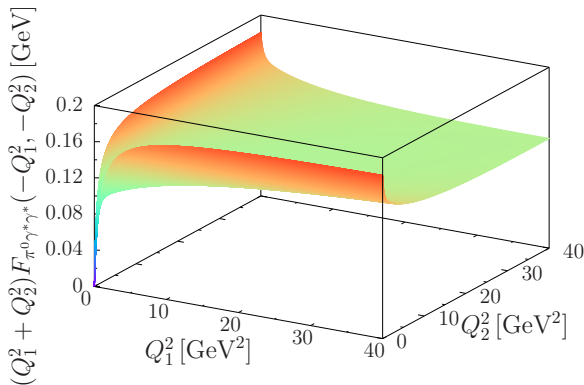
Diagonal form factor in $Q^2 F_{\pi^0 \gamma^* \gamma^*}(-Q^2, -Q^2)$ in comparison to LMD+V fit to lattice:

Gérardin et al., 2016



Numerical results

$(Q_1^2 + Q_2^2)F_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$ as a function of Q_1^2 and Q_2^2 :



- $1/Q_i^2$ behavior in the **entire domain** of space-like virtualities
⇒ Hard to obtain in resonance models

Numerical results

Pion-pole contribution to a_μ from the final representation:

$$\begin{aligned} a_\mu^{\pi^0\text{-pole}} &= 63.0(0.9)_{F_{\pi\gamma\gamma}}(1.1)_{\text{disp}}(2.2)_{\text{BL}}(0.6)_{\text{asym}} \times 10^{-11} \\ &= 63.0^{+2.7}_{-2.1} \times 10^{-11} \end{aligned}$$

- First complete data-driven determination
- Fully controlled uncertainty estimates

Comparison to lattice:

- Lattice 2016: $(65.0 \pm 8.3) \times 10^{-11}$ Gérardin et al., 2016
- Lattice 2019: $(59.7 \pm 3.6) \times 10^{-11}$, changes to $(62.3 \pm 2.3) \times 10^{-11}$ after fixing normalization from experiment Gérardin et al., 2019

Conclusion and outlook

- Hadronic vacuum polarization: 3π
 - ▶ Independent dispersive analysis for the 3π channel
 - ▶ **Resolved** main tension in 3π , **reaffirmed** tension in a_μ
- Hadronic light-by-light scattering: pion pole
 - ▶ Dispersive reconstruction of the pion transition form factor
 - ▶ Data-driven determination of $a_\mu^{\pi^0\text{-pole}}$ with **carefully estimated improvable** uncertainties
- Similar analysis done for $\pi^0\gamma$ Hoid et al. 2020
- $\pi^0 \rightarrow e^+e^-$ Hoferichter et al. 2021
- η and η' pole contributions S. Holz et al.

Much obliged for your attention!

