第六届强子谱和强子结构研讨会

中国科学院大学
University of Chinese Academy of Sciences

Covariant L-S coupling scheme

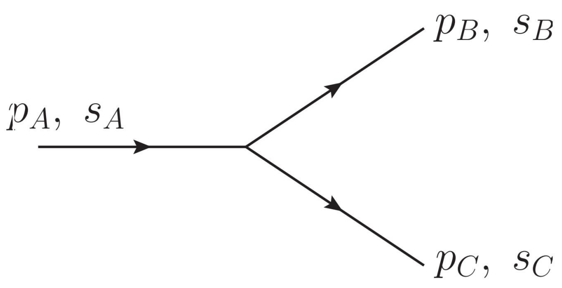
Hao-Jie Jing

2023-08-28, Beijing

Based on : *Covariant orbital-spin scheme for any spin based on irreducible tensor*
H.J.Jing, Di Ben, Shu-Ming Wu, Jia-Jun Wu and Bing-Song Zou. 10.1007/JHEP06(2023)039
Partial wave analysis of decays with arbitrary spins
Xiao-Yu Li, Xiang-Kun Dong and **H.J.Jing**. To be published on NPA (arXiv: 2212.06417)

Review of covariant L-S scheme

Building blocks: **B.S.Zou and D.V.Bugg, Eur.Phys.J.A,16,537-547 (2003)**



- Spin wave function for bosons $\phi_{\mu_1\cdots\mu_S}$
- Pure spin wave function for fermion pairs

B.S.Zou and F.Hussain, Phys.Rev.C.67.015204 (2003)

$$\psi^{(n)}_{\mu_1\cdots\mu_n}=\overline{u}_{\mu_1\cdots\mu_n}(p_B,s_B)\gamma_5 v(p_C,s_C)$$

$$\Psi^{(n+1)}_{\mu_1\cdots\mu_{n+1}}=\overline{u}_{\mu_1\cdots\mu_n}(p_B,s_B)\left(\gamma_{\mu_{n+1}}-\frac{r_{\mu_{n+1}}}{m_A+m_B+m_C}\right)v(p_C,s_C)+(\mu_1\leftrightarrow\mu_{n+1})+\cdots+(\mu_n\leftrightarrow\mu_{n+1})$$

$$\phi^{(n)}_{\mu_1\cdots\mu_n}=\overline{u}(p_B,s_B)u_{\mu_1\cdots\mu_n}(p_A,s_A)$$

$$\Phi^{(n+1)}_{\mu_1\cdots\mu_{n+1}}=\overline{u}(p_B,s_B)\gamma_5\tilde{\gamma}_{\mu_{n+1}}u_{\mu_1\cdots\mu_n}(p_A,s_A)+(\mu_1\leftrightarrow\mu_{n+1})+\cdots+(\mu_n\leftrightarrow\mu_{n+1})$$

$$r=p_B-p_C$$

$$-g_{\mu\nu}+\frac{p_\mu p_\nu}{p^2}\equiv-\tilde{g}_{\mu\nu}(p)$$

$$\tilde{\gamma}_\mu=\tilde{g}_{\mu\nu}(p_A)\gamma^\nu$$

- Orbital angular momentum tensor $\tilde{t}^{(L)}_{\mu_1\cdots\mu_L}$
- Lorentz structures $(p_A)_\mu,\ g_{\mu\nu},\ \epsilon_{\mu\nu\rho\sigma}$

For radiative decay process:

S.Dulat and B.S.Zou, Eur.Phys.J.A, 26,125-134 (2005)
S.Dulat, J.J.Wu and B.S.Zou, PhysRevD.83.094032 (2011)

- additional conditions on amplitudes due to gauge invariance

Spin wave function for massive particle

A massive particle can be stationary $p_\mu = \Lambda_\mu{}^\nu\, k_\nu \ \left[\Lambda = R \cdot B_z \cdot R^{-1},\ k_\mu = (m, 0, 0, 0)_\mu\right]$

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$$L_p \longrightarrow \mathrm{SU}(2) \longrightarrow \mathrm{U}(1)$$

$$C_{L/R} = (J_1 \pm iK_1)^2 + (J_2 \pm iK_2)^2 + (J_3 \pm iK_3)^2$$

$$C_{\mathrm{SU}(2)} = J_1^2 + J_2^2 + J_3^2 \qquad C_{\mathrm{U}(1)} = J_3$$

J.Q.Chen, M.J.Gao, and G.Q.Ma, Rev.Mod.Phys.57,211(1985)

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$$\left(C_{L/R}\right)_\alpha{}^\beta\ u^\sigma_\beta(s) = s_{L/R}\left(s_{L/R}+1\right)\ u^\sigma_\alpha(s)$$

$$\left(C_{\mathrm{SU}(2)}\right)_\alpha{}^\beta\ u^\sigma_\beta(s) = s(s+1)\ u^\sigma_\alpha(s)$$

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$$D_\alpha{}^\beta(R)\; u_\beta^\sigma(s) = u_\beta^{\sigma'}(s)\; D_{\sigma'}^{(s)\sigma}(R)$$

$$D_\alpha{}^\beta(R)\; u_\beta^{\sigma'}(s)\; D_{\sigma'}^{(s)\sigma}\left(R^{-1}\right) = u_\alpha^\sigma(s)$$

S.Weinberg, Phys.Rev.133.B1318(1964)

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$$u_\alpha^\sigma(\mathbf{p}, s) \equiv D_\alpha{}^\beta(\Lambda)\ u_\beta^\sigma(s)$$

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→

$$u_\alpha^\sigma(\mathbf{p}, s) \equiv D_\alpha{}^\beta(\Lambda) \; u_\beta^\sigma(s)$$

↓

$$P_\beta^{\alpha_1\alpha_2}(\mathbf{p}; \chi, s) = u_\beta^\sigma(\mathbf{p}; s) \, \bar{u}_\sigma^{\alpha_1\alpha_2}(\mathbf{p}; \chi^*, s)$$

Ir.ten.s of SO(3)

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↓

$$P_\beta^{\alpha_1\alpha_2}(\mathbf{p};\chi,s) = u_\beta^\sigma(\mathbf{p};s)\ \bar{u}_\sigma^{\alpha_1\alpha_2}(\mathbf{p};\chi^*,s)$$

Ir.ten.s of SO(3)

←

$$T_\beta^{\alpha_1\alpha_2} = \sum_{s\in[\beta]} u_\beta^\sigma(s)\ \bar{u}_\sigma^{\alpha_1\alpha_2}([\beta]^*,s)$$

Ir.ten.s of the Lorentz group

Partial wave decomposition

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = \Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)\ \bar{u}_{\sigma_1}^{\alpha_1}(\mathbf{p}_1, s_1)\, u_{\alpha_2}^{\sigma_2}(\mathbf{p}_2, s_2)\, u_{\alpha_3}^{\sigma_3}(\mathbf{p}_3, s_3)$$

Partial wave decomposition

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}\left(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\right)=\Gamma_{\alpha_1}^{\alpha_2\alpha_3}\left(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\right)\;\bar{u}_{\sigma_1}^{\alpha_1}(\mathbf{p}_1,s_1)\,u_{\alpha_2}^{\sigma_2}(\mathbf{p}_2,s_2)\,u_{\alpha_3}^{\sigma_3}(\mathbf{p}_3,s_3)$$

- **Definition of pure-orbital and pure-spin component**

Partial wave decomposition

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$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*) = \underbrace{\Gamma_{\alpha_1}^{\beta_2\beta_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)\,D_{\beta_2}{}^{\alpha_2}(\Lambda_{2^*})\ D_{\beta_3}{}^{\alpha_3}(\Lambda_{3^*})}_{\text{pure-orbital part}} \times \underbrace{\bar{u}_{\sigma_1}^{\alpha_1}(s_1)\,u_{\alpha_2}^{\sigma_2}(s_2)\,u_{\alpha_3}^{\sigma_3}(s_3)}_{\text{pure-spin part}}$$

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- **Decomposition of covariant tensors**

Partial wave decomposition

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- **Decomposition of covariant tensors**

$$O_{\alpha_1}^{\alpha_2\alpha_3} = \sum_{[\alpha]} T_{\alpha_1}^{\alpha_2\alpha_3\alpha}\left(O^{[\alpha]}\right)_{\alpha}$$

Partial wave decomposition

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- **Decomposition of covariant tensors**

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Partial wave decomposition

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)=\Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)\;\bar{u}_{\sigma_1}^{\alpha_1}(\mathbf{p}_1,s_1)\,u_{\alpha_2}^{\sigma_2}(\mathbf{p}_2,s_2)\,u_{\alpha_3}^{\sigma_3}(\mathbf{p}_3,s_3)$$

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- **Decomposition of covariant tensors**

$$O_{\alpha_1}^{\alpha_2\alpha_3}=\sum_{[\alpha]}T_{\alpha_1}^{\alpha_2\alpha_3\alpha}\left(O^{[\alpha]}\right)_\alpha \quad\Rightarrow\quad \left(O^{[\alpha]}\right)_\alpha=\sum_{s\in[\alpha]}P_\alpha{}^\beta(\mathbf{p},s)\left(O_s^{[\alpha]}\right)_\beta$$

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)=\sum_{(L,S)}c_{(L,S)}\,\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S)$$

Partial wave decomposition

- **Example: amplitude with spin-1/2**

Lorentz structure	Partial wave components $\left({}^{2S+1}L_J\right)$
$\bar{\psi}_2\psi_1$	${}^3P_0$ ${}^1S_0$
$\bar{\psi}_2\gamma_5\psi_1$	${}^1S_0$ ${}^3P_0$
$\bar{\psi}_2\gamma_\mu\psi_1$	${}^3P_0$ ${}^3S_1$ ${}^3D_1$ ${}^1S_0$ ${}^1P_1$ ${}^3P_1$
$\bar{\psi}_2\gamma_5\gamma_\mu\psi_1$	${}^3P_1$ ${}^1S_0$ ${}^1P_1$ ${}^3P_0$ ${}^3S_1$ ${}^3D_1$
$\bar{\psi}_2\sigma_{\mu\nu}\psi_1$	${}^1P_1$ ${}^3S_1$ ${}^3P_1$ ${}^3D_1$ ${}^1P_1$ ${}^3S_1$ ${}^3P_1$ ${}^3D_1$
$\bar{\psi}_2 \overleftrightarrow{\partial}_\mu \psi_1$	${}^3S_1$ ${}^3P_0$ ${}^3D_1$ ${}^1S_0$
$\partial_\mu\left(\bar{\psi}_2\psi_1\right)$	${}^3P_0$ ${}^1S_0$ ${}^1P_1$
$\vdots$	$\vdots$

Partial wave decomposition

- **Example: amplitude with spin-1/2**

$\bar{\psi}_2\gamma_\mu\psi_1$ $\quad$ ${}^3P_0\ {}^3S_1\ {}^3D_1$

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$\bar{\psi}_2\gamma_5\gamma_\mu\psi_1$	${}^3P_1\ {}^1S_0\ {}^1P_1$ ${}^3P_0\ {}^3S_1\ {}^3D_1$
$\bar{\psi}_2\sigma_{\mu\nu}\psi_1$	${}^1P_1\ {}^3S_1\ {}^3P_1\ {}^3D_1$ ${}^1P_1\ {}^3S_1\ {}^3P_1\ {}^3D_1$
$\bar{\psi}_2 \overleftrightarrow{\partial}_\mu \psi_1$	${}^3S_1\ {}^3P_0\ {}^3D_1$ ${}^1S_0$
$\partial_\mu\left(\bar{\psi}_2\psi_1\right)$	${}^3P_0$ ${}^1S_0\ {}^1P_1$
$\vdots$	$\vdots$

Partial wave decomposition

- **Example: amplitude with spin-1/2**

$\bar{\psi}_2\gamma_\mu\psi_1$ $\quad$ $\cancel{^3P_0}$ $^3S_1$ $^3D_1$

$\bar{\psi}_2 \overleftrightarrow{\partial}_\mu \psi_1$ $\quad$ $^3S_1$ $\cancel{^3P_0}$ $^3D_1$

$$V \to B\bar{B}$$

$$V^\mu \cdot \bar{\psi}_2 \left(c_1\gamma_\mu + c_2 \overleftrightarrow{\partial}_\mu \right) \psi_1 \quad (^3S_1/^3D_1)$$

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$\bar{\psi}_2\gamma_5\gamma_\mu\psi_1$	$^3P_1$ $^1S_0$ $^1P_1$ $^3P_0$ $^3S_1$ $^3D_1$
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$\bar{\psi}_2 \overleftrightarrow{\partial}_\mu \psi_1$	$^3S_1$ $^3P_0$ $^3D_1$ $^1S_0$
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$\vdots$	$\vdots$

Partial wave decomposition

- **Example: amplitude with spin-1/2**

$\bar{\psi}_2\gamma_\mu\psi_1$ ~~$^3P_0$~~ $^3S_1$ $^3D_1$

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$$V^\mu \cdot \bar{\psi}_2 \left(c_1\gamma_\mu + c_2 \overset{\leftrightarrow}{\partial}_\mu\right) \psi_1 \quad (^3S_1/^3D_1)$$

- **It is possible to obtain partial wave amplitudes through the linear combination of various Lorentz structures.**

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$\bar{\psi}_2\sigma_{\mu\nu}\psi_1$	$^1P_1$ $^3S_1$ $^3P_1$ $^3D_1$ $^1P_1$ $^3S_1$ $^3P_1$ $^3D_1$
$\bar{\psi}_2 \overset{\leftrightarrow}{\partial}_\mu \psi_1$	$^3S_1$ $^3P_0$ $^3D_1$ $^1S_0$
$\partial_\mu\left(\bar{\psi}_2\psi_1\right)$	$^3P_0$ $^1S_0$ $^1P_1$
$\vdots$	$\vdots$

Partial wave decomposition

- **Example: amplitude with spin-3/2**

Lorentz structure	Partial wave components $\left({}^{2S+1}L_J\right)$
$\bar{\psi}_2\psi_{1\mu}$	${}^5D_0\ {}^3P_1\ {}^5P_1\ {}^5F_1$ ${}^3P_0\ {}^3S_1\ {}^3D_1$
$\bar{\psi}_2\gamma_5\psi_{1\mu}$	${}^3P_0\ {}^3S_1\ {}^3D_1$ ${}^5D_0\ {}^3P_1\ {}^5P_1\ {}^5F_1$
$\bar{\psi}_2\gamma_\nu\psi_{1\mu}$	${}^5D_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^5S_2\ {}^3D_2\ {}^5D_2$ ${}^3P_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^3P_2\ {}^5P_2$
$\bar{\psi}_2\gamma_5\gamma_\nu\psi_{1\mu}$	${}^3P_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^3P_2\ {}^5P_2$ ${}^5D_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^5S_2\ {}^3D_2\ {}^5D_2$
$\bar{\psi}_2\sigma_{\nu\rho}\psi_{1\mu}$	${}^3P_0\ {}^5D_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^5S_2\ {}^3P_2\ {}^5P_2\ {}^3D_2\ {}^5D_2$ ${}^3P_0\ {}^5D_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^5S_2\ {}^3P_2\ {}^5P_2\ {}^3D_2\ {}^5D_2$
$\bar{\psi}_2\ \overset{\leftrightarrow}{\partial}_\nu\ \psi_{1\mu}$	${}^5D_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^5F_1\ {}^5S_2\ {}^3D_2\ {}^5D_2\ {}^5G_2$ ${}^3P_0\ {}^3S_1\ {}^3D_1$
$\partial_\nu\left(\bar{\psi}_2\psi_{1\mu}\right)$	${}^5D_0\ {}^3P_1\ {}^5P_1\ {}^5F_1$ ${}^3P_0\ {}^3S_1\ {}^3P_1\ {}^3D_1\ {}^3P_2\ {}^3F_2$
$\vdots$	$\vdots$

Partial wave decomposition

- **Example: amplitude with spin-3/2**

- **With the increase of spin, the amount of computation will increase dramatically.**

Lorentz structure	Partial wave components $\left({}^{2S+1}L_J\right)$
$\bar{\psi}_2\psi_{1\mu}$	${}^5D_0\ {}^3P_1\ {}^5P_1\ {}^5F_1$ ${}^3P_0\ {}^3S_1\ {}^3D_1$
$\bar{\psi}_2\gamma_5\psi_{1\mu}$	${}^3P_0\ {}^3S_1\ {}^3D_1$ ${}^5D_0\ {}^3P_1\ {}^5P_1\ {}^5F_1$
$\bar{\psi}_2\gamma_\nu\psi_{1\mu}$	${}^5D_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^5S_2\ {}^3D_2\ {}^5D_2$ ${}^3P_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^3P_2\ {}^5P_2$
$\bar{\psi}_2\gamma_5\gamma_\nu\psi_{1\mu}$	${}^3P_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^3P_2\ {}^5P_2$ ${}^5D_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^5S_2\ {}^3D_2\ {}^5D_2$
$\bar{\psi}_2\sigma_{\nu\rho}\psi_{1\mu}$	${}^3P_0\ {}^5D_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^5S_2\ {}^3P_2\ {}^5P_2\ {}^3D_2\ {}^5D_2$ ${}^3P_0\ {}^5D_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^5S_2\ {}^3P_2\ {}^5P_2\ {}^3D_2\ {}^5D_2$
$\bar{\psi}_2\ \overleftrightarrow{\partial}_\nu\ \psi_{1\mu}$	${}^5D_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^5F_1\ {}^5S_2\ {}^3D_2\ {}^5D_2\ {}^5G_2$ ${}^3P_0\ {}^3S_1\ {}^3D_1$
$\partial_\nu\left(\bar{\psi}_2\psi_{1\mu}\right)$	${}^5D_0\ {}^3P_1\ {}^5P_1\ {}^5F_1$ ${}^3P_0\ {}^3S_1\ {}^3P_1\ {}^3D_1\ {}^3P_2\ {}^3F_2$
$\vdots$	$\vdots$

Partial wave decomposition

- Example: amplitude with spin-3/2

- With the increase of spin, the amount of computation will in-crease dramatically.

- It is useful to consider how to construct the Lorentz covariant partial wave amplitude.

Lorentz structure	Partial wave components $\left({}^{2S+1}L_J\right)$
$\bar{\psi}_2\psi_{1\mu}$	${}^5D_0\ {}^3P_1\ {}^5P_1\ {}^5F_1$ ${}^3P_0\ {}^3S_1\ {}^3D_1$
$\bar{\psi}_2\gamma_5\psi_{1\mu}$	${}^3P_0\ {}^3S_1\ {}^3D_1$ ${}^5D_0\ {}^3P_1\ {}^5P_1\ {}^5F_1$
$\bar{\psi}_2\gamma_\nu\psi_{1\mu}$	${}^5D_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^5S_2\ {}^3D_2\ {}^5D_2$ ${}^3P_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^3P_2\ {}^5P_2$
$\bar{\psi}_2\gamma_5\gamma_\nu\psi_{1\mu}$	${}^3P_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^3P_2\ {}^5P_2$ ${}^5D_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^5S_2\ {}^3D_2\ {}^5D_2$
$\bar{\psi}_2\sigma_{\nu\rho}\psi_{1\mu}$	${}^3P_0\ {}^5D_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^5S_2\ {}^3P_2\ {}^5P_2\ {}^3D_2\ {}^5D_2$ ${}^3P_0\ {}^5D_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^5S_2\ {}^3P_2\ {}^5P_2\ {}^3D_2\ {}^5D_2$
$\bar{\psi}_2\overleftrightarrow{\partial}_\nu\psi_{1\mu}$	${}^5D_0\ {}^3S_1\ {}^3P_1\ {}^5P_1\ {}^3D_1\ {}^5D_1\ {}^5F_1\ {}^5S_2\ {}^3D_2\ {}^5D_2\ {}^5G_2$ ${}^3P_0\ {}^3S_1\ {}^3D_1$
$\partial_\nu\left(\bar{\psi}_2\psi_{1\mu}\right)$	${}^5D_0\ {}^3P_1\ {}^5P_1\ {}^5F_1$ ${}^3P_0\ {}^3S_1\ {}^3P_1\ {}^3D_1\ {}^3P_2\ {}^3F_2$
$\vdots$	$\vdots$

Construction of partial wave amplitudes

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*) = \underbrace{\Gamma_{\alpha_1}^{\beta_2\beta_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)\, D_{\beta_2}^{\ \ \alpha_2}(\Lambda_{2^*})\ D_{\beta_3}^{\ \ \alpha_3}(\Lambda_{3^*})}_{\text{pure-orbital part}} \times \underbrace{\bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2}(s_2)\, u_{\alpha_3}^{\sigma_3}(s_3)}_{\text{pure-spin part}}$$

Construction of partial wave amplitudes

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*) = \underbrace{\Gamma_{\alpha_1}^{\beta_2\beta_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)\, D_{\beta_2}{}^{\alpha_2}(\Lambda_{2^*})\; D_{\beta_3}{}^{\alpha_3}(\Lambda_{3^*})}_{\text{pure-orbital part}} \times \underbrace{\bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2}(s_2)\, u_{\alpha_3}^{\sigma_3}(s_3)}_{\text{pure-spin part}}$$

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S) = \Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S)\; \bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2}(s_2)\, u_{\alpha_3}^{\sigma_3}(s_3)$$

$$\Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S) = P_{\alpha_1}^{\alpha_L\alpha_S}(\mathbf{k}_1;\chi_{LS},s_1)\, P_{\alpha_S}^{\alpha_2\alpha_3}(\mathbf{k}_1;\chi_{23},S)\, \tilde{t}_{\alpha_L}^{(L)}(\mathbf{k}_1,\mathbf{p}_2^*-\mathbf{p}_3^*)$$

Construction of partial wave amplitudes

- **Partial wave formula A**

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*) = \underbrace{\Gamma_{\alpha_1}^{\beta_2\beta_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)\, D_{\beta_2}{}^{\alpha_2}(\Lambda_{2^*})\, D_{\beta_3}{}^{\alpha_3}(\Lambda_{3^*})}_{\text{pure-orbital part}} \times \underbrace{\bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2}(s_2)\, u_{\alpha_3}^{\sigma_3}(s_3)}_{\text{pure-spin part}}$$

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S) = \Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S)\; \bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2}(s_2)\, u_{\alpha_3}^{\sigma_3}(s_3)$$

Consistent with the helicity amplitude. **M.Jacob and G.C.Wick, Annals of Physics,7,4,404-428(1959)**

$$\Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S) = P_{\alpha_1}^{\alpha_L\alpha_S}(\mathbf{k}_1;\chi_{LS},s_1)\, P_{\alpha_S}^{\alpha_2\alpha_3}(\mathbf{k}_1;\chi_{23},S)\, \tilde{t}_{\alpha_L}^{(L)}(\mathbf{k}_1,\mathbf{p}_2^*-\mathbf{p}_3^*)$$

Construction of partial wave amplitudes

- **Partial wave formula A**

$$\mathcal{A}^{\sigma_2\sigma_3}_{\sigma_1}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*) = \underbrace{\Gamma^{\beta_2\beta_3}_{\alpha_1}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)\, D_{\beta_2}{}^{\alpha_2}(\Lambda_{2^*})\, D_{\beta_3}{}^{\alpha_3}(\Lambda_{3^*})}_{\text{pure-orbital part}} \times \underbrace{\bar{u}^{\alpha_1}_{\sigma_1}(s_1)\, u^{\sigma_2}_{\alpha_2}(s_2)\, u^{\sigma_3}_{\alpha_3}(s_3)}_{\text{pure-spin part}}$$

$$\mathcal{A}^{\sigma_2\sigma_3}_{\sigma_1}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S) = \Gamma^{\alpha_2\alpha_3}_{\alpha_1}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S)\,\bar{u}^{\alpha_1}_{\sigma_1}(s_1)\, u^{\sigma_2}_{\alpha_2}(s_2)\, u^{\sigma_3}_{\alpha_3}(s_3)$$

Consistent with the helicity amplitude. **M.Jacob and G.C.Wick, Annals of Physics,7,4,404-428(1959)**

$$\mathcal{B}^{\sigma_2\sigma_3}_{\sigma_1}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*) = \underbrace{\Gamma^{\alpha_2\alpha_3}_{\alpha_1}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)}_{\text{pure-orbital part}} \times \underbrace{\bar{u}^{\alpha_1}_{\sigma_1}(s_1)\, u^{\sigma_2}_{\alpha_2}(\mathbf{p}_2^*,s_2)\, u^{\sigma_3}_{\alpha_3}(\mathbf{p}_3^*,s_3)}_{\text{pure-spin part}}$$

$$\Gamma^{\alpha_2\alpha_3}_{\alpha_1}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S) = P^{\alpha_L\alpha_S}_{\alpha_1}(\mathbf{k}_1;\chi_{LS},s_1)\, P^{\alpha_2\alpha_3}_{\alpha_S}(\mathbf{k}_1;\chi_{23},S)\, \tilde{t}^{(L)}_{\alpha_L}(\mathbf{k}_1,\mathbf{p}_2^*-\mathbf{p}_3^*)$$

Construction of partial wave amplitudes

- **Partial wave formula A**

$$\mathcal{A}^{\sigma_2\sigma_3}_{\sigma_1}(\mathbf{k}_1,\mathbf{p}^*_2,\mathbf{p}^*_3) = \underbrace{\Gamma^{\beta_2\beta_3}_{\alpha_1}(\mathbf{k}_1,\mathbf{p}^*_2,\mathbf{p}^*_3)\, D_{\beta_2}{}^{\alpha_2}(\Lambda_{2^*})\; D_{\beta_3}{}^{\alpha_3}(\Lambda_{3^*})}_{\text{pure-orbital part}} \times \underbrace{\bar{u}^{\alpha_1}_{\sigma_1}(s_1)\, u^{\sigma_2}_{\alpha_2}(s_2)\, u^{\sigma_3}_{\alpha_3}(s_3)}_{\text{pure-spin part}}$$

$$\mathcal{A}^{\sigma_2\sigma_3}_{\sigma_1}(\mathbf{k}_1,\mathbf{p}^*_2,\mathbf{p}^*_3;L,S) = \Gamma^{\alpha_2\alpha_3}_{\alpha_1}(\mathbf{k}_1,\mathbf{p}^*_2,\mathbf{p}^*_3;L,S)\;\bar{u}^{\alpha_1}_{\sigma_1}(s_1)\, u^{\sigma_2}_{\alpha_2}(s_2)\, u^{\sigma_3}_{\alpha_3}(s_3)$$

Consistent with the helicity amplitude. **M.Jacob and G.C.Wick, Annals of Physics,7,4,404-428(1959)**

$$\mathcal{B}^{\sigma_2\sigma_3}_{\sigma_1}(\mathbf{k}_1,\mathbf{p}^*_2,\mathbf{p}^*_3) = \underbrace{\Gamma^{\alpha_2\alpha_3}_{\alpha_1}(\mathbf{k}_1,\mathbf{p}^*_2,\mathbf{p}^*_3)}_{\text{pure-orbital part}} \times \underbrace{\bar{u}^{\alpha_1}_{\sigma_1}(s_1)\, u^{\sigma_2}_{\alpha_2}(\mathbf{p}^*_2,s_2)\, u^{\sigma_3}_{\alpha_3}(\mathbf{p}^*_3,s_3)}_{\text{pure-spin part}}$$

$$\mathcal{B}^{\sigma_2\sigma_3}_{\sigma_1}(\mathbf{k}_1,\mathbf{p}^*_2,\mathbf{p}^*_3;L,S) = \Gamma^{\alpha_2\alpha_3}_{\alpha_1}(\mathbf{k}_1,\mathbf{p}^*_2,\mathbf{p}^*_3;L,S)\;\bar{u}^{\alpha_1}_{\sigma_1}(s_1)\, u^{\sigma_2}_{\alpha_2}(\mathbf{p}^*_2,s_2)\, u^{\sigma_3}_{\alpha_3}(\mathbf{p}^*_3,s_3)$$

$$\Gamma^{\alpha_2\alpha_3}_{\alpha_1}(\mathbf{k}_1,\mathbf{p}^*_2,\mathbf{p}^*_3;L,S) = P^{\alpha_L\alpha_S}_{\alpha_1}(\mathbf{k}_1;\chi_{LS},s_1)\,P^{\alpha_2\alpha_3}_{\alpha_S}(\mathbf{k}_1;\chi_{23},S)\,\tilde{t}^{(L)}_{\alpha_L}(\mathbf{k}_1,\mathbf{p}^*_2-\mathbf{p}^*_3)$$

Construction of partial wave amplitudes

- **Partial wave formula A**

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*) = \underbrace{\Gamma_{\alpha_1}^{\beta_2\beta_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)\, D_{\beta_2}{}^{\alpha_2}(\Lambda_{2^*})\; D_{\beta_3}{}^{\alpha_3}(\Lambda_{3^*})}_{\text{pure-orbital part}} \times \underbrace{\bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2}(s_2)\, u_{\alpha_3}^{\sigma_3}(s_3)}_{\text{pure-spin part}}$$

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S) = \Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S)\; \bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2}(s_2)\, u_{\alpha_3}^{\sigma_3}(s_3)$$

Consistent with the helicity amplitude. **M.Jacob and G.C.Wick, Annals of Physics,7,4,404-428(1959)**

- **Partial wave formula B**

$$\mathcal{B}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*) = \underbrace{\Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)}_{\text{pure-orbital part}} \times \underbrace{\bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2}(\mathbf{p}_2^*,s_2)\, u_{\alpha_3}^{\sigma_3}(\mathbf{p}_3^*,s_3)}_{\text{pure-spin part}}$$

$$\mathcal{B}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S) = \Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S)\; \bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2}(\mathbf{p}_2^*,s_2)\, u_{\alpha_3}^{\sigma_3}(\mathbf{p}_3^*,s_3)$$

Consistent with the covariant tensor amplitude. **S.U.Chung, Phys.Rev.D.57.431-442(1998)**

$$\Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S) = P_{\alpha_1}^{\alpha_L\alpha_S}(\mathbf{k}_1;\chi_{LS},s_1)\, P_{\alpha_S}^{\alpha_2\alpha_3}(\mathbf{k}_1;\chi_{23},S)\, \tilde{t}_{\alpha_L}^{(L)}(\mathbf{k}_1,\mathbf{p}_2^*-\mathbf{p}_3^*)$$

Construction of partial wave amplitudes

- **Partial wave formula A**

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*) = \underbrace{\Gamma_{\alpha_1}^{\beta_2\beta_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)\, D_{\beta_2}{}^{\alpha_2}(\Lambda_{2^*})\; D_{\beta_3}{}^{\alpha_3}(\Lambda_{3^*})}_{\text{pure-orbital part}} \times \underbrace{\bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2}(s_2)\, u_{\alpha_3}^{\sigma_3}(s_3)}_{\text{pure-spin part}}$$

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S) = \Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S)\; \bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2}(s_2)\, u_{\alpha_3}^{\sigma_3}(s_3)$$

Consistent with the helicity amplitude. M.Jacob and G.C.Wick, Annals of Physics,7,4,404-428(1959)

- **Partial wave formula B**

$$\mathcal{B}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*) = \underbrace{\Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)}_{\text{pure-orbital part}} \times \underbrace{\bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2}(\mathbf{p}_2^*,s_2)\, u_{\alpha_3}^{\sigma_3}(\mathbf{p}_3^*,s_3)}_{\text{pure-spin part}}$$

$$\mathcal{B}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S) = \Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S)\; \bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2}(\mathbf{p}_2^*,s_2)\, u_{\alpha_3}^{\sigma_3}(\mathbf{p}_3^*,s_3)$$

Consistent with the covariant tensor amplitude. S.U.Chung, Phys.Rev.D.57.431-442(1998)

$$\Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S) = P_{\alpha_1}^{\alpha_L\alpha_S}(\mathbf{k}_1;\chi_{LS},s_1)\, P_{\alpha_S}^{\alpha_2\alpha_3}(\mathbf{k}_1;\chi_{23},S)\, \tilde{t}_{\alpha_L}^{(L)}(\mathbf{k}_1,\mathbf{p}_2^*-\mathbf{p}_3^*)$$

- **From c.m.f. to any frame**

Construction of partial wave amplitudes

- **Partial wave formula A**

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*) = \underbrace{\Gamma_{\alpha_1}^{\beta_2\beta_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)\, D_{\beta_2}{}^{\alpha_2}(\Lambda_{2^*})\; D_{\beta_3}{}^{\alpha_3}(\Lambda_{3^*})}_{\text{pure-orbital part}} \times \underbrace{\bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2}(s_2)\, u_{\alpha_3}^{\sigma_3}(s_3)}_{\text{pure-spin part}}$$

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S) = \Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S)\; \bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2}(s_2)\, u_{\alpha_3}^{\sigma_3}(s_3)$$

Consistent with the helicity amplitude. **M.Jacob and G.C.Wick, Annals of Physics,7,4,404-428(1959)**

- **Partial wave formula B**

$$\mathcal{B}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*) = \underbrace{\Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)}_{\text{pure-orbital part}} \times \underbrace{\bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2}(\mathbf{p}_2^*,s_2)\, u_{\alpha_3}^{\sigma_3}(\mathbf{p}_3^*,s_3)}_{\text{pure-spin part}}$$

$$\mathcal{B}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S) = \Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S)\; \bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2}(\mathbf{p}_2^*,s_2)\, u_{\alpha_3}^{\sigma_3}(\mathbf{p}_3^*,s_3)$$

Consistent with the covariant tensor amplitude. **S.U.Chung, Phys.Rev.D.57.431-442(1998)**

$$\Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S) = P_{\alpha_1}^{\alpha_L\alpha_S}(\mathbf{k}_1;\chi_{LS},s_1)\, P_{\alpha_S}^{\alpha_2\alpha_3}(\mathbf{k}_1;\chi_{23},S)\, \tilde{t}_{\alpha_L}^{(L)}(\mathbf{k}_1,\mathbf{p}_2^*-\mathbf{p}_3^*)$$

- **From c.m.f. to any frame** $D_{\alpha}{}^{\beta}(\Lambda_1)\; u_{\beta}^{\sigma}(\mathbf{p}_i^*,s_i) = u_{\alpha}^{\sigma'}(\mathbf{p}_i,s_i)\; D_{\sigma'}^{(s_i)\sigma}(R_{1i}) \quad [R_{1i} = \Lambda_i^{-1}\cdot\Lambda_1\cdot\Lambda_{i^*}] \quad (i=2,3)$

Construction of partial wave amplitudes

- **Partial wave formula A**

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*) = \underbrace{\Gamma_{\alpha_1}^{\beta_2\beta_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)\, D_{\beta_2}{}^{\alpha_2}(\Lambda_{2^*})\; D_{\beta_3}{}^{\alpha_3}(\Lambda_{3^*})}_{\text{pure-orbital part}} \times \underbrace{\bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2}(s_2)\, u_{\alpha_3}^{\sigma_3}(s_3)}_{\text{pure-spin part}}$$

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S) = \Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S)\;\bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2}(s_2)\, u_{\alpha_3}^{\sigma_3}(s_3)$$

Consistent with the helicity amplitude. **M.Jacob and G.C.Wick, Annals of Physics,7,4,404-428(1959)**

- **Partial wave formula B**

$$\mathcal{B}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*) = \underbrace{\Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)}_{\text{pure-orbital part}} \times \underbrace{\bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2}(\mathbf{p}_2^*,s_2)\, u_{\alpha_3}^{\sigma_3}(\mathbf{p}_3^*,s_3)}_{\text{pure-spin part}}$$

$$\mathcal{B}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S) = \Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S)\;\bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2}(\mathbf{p}_2^*,s_2)\, u_{\alpha_3}^{\sigma_3}(\mathbf{p}_3^*,s_3)$$

Consistent with the covariant tensor amplitude. **S.U.Chung, Phys.Rev.D.57.431-442(1998)**

$$\Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*;L,S) = P_{\alpha_1}^{\alpha_L\alpha_S}(\mathbf{k}_1;\chi_{LS},s_1)\, P_{\alpha_S}^{\alpha_2\alpha_3}(\mathbf{k}_1;\chi_{23},S)\,\tilde{t}_{\alpha_L}^{(L)}(\mathbf{k}_1,\mathbf{p}_2^*-\mathbf{p}_3^*)$$

- **From c.m.f. to any frame** $D_\alpha{}^\beta(\Lambda_1)\; u_\beta^\sigma(\mathbf{p}_i^*,s_i) = u_\alpha^{\sigma'}(\mathbf{p}_i,s_i)\; D_{\sigma'}^{(s_i)\sigma}(R_{1i})\quad [R_{1i} = \Lambda_i^{-1}\cdot\Lambda_1\cdot\Lambda_{i^*}]\quad (i=2,3)$

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Construction of partial wave amplitudes

- **Partial wave formula A**

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*) = \underbrace{\Gamma_{\alpha_1}^{\beta_2\beta_3}(\mathbf{k}_1,\mathbf{p}_2^*,\mathbf{p}_3^*)\, D_{\beta_2}{}^{\alpha_2}(\Lambda_{2^*})\ D_{\beta_3}{}^{\alpha_3}(\Lambda_{3^*})}_{\text{pure-orbital part}} \times \underbrace{\bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2}(s_2)\, u_{\alpha_3}^{\sigma_3}(s_3)}_{\text{pure-spin part}}$$

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Spin wave function for massless particle

A massless particle can not be stationary $p_\mu = \tilde{\Lambda}_\mu{}^\nu\, k_\nu \;\left[\tilde{\Lambda} = R \cdot B_z,\; k_\mu = (\kappa, 0, 0, \kappa)_\mu\right]$

S.Weinberg, PhysRev.134.B882(1964)

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$$L_p \quad \longrightarrow \quad \mathrm{E}(2) \quad \longrightarrow \quad \mathrm{U}(1)$$

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$$h^\sigma_\beta(s,\tilde{s}) \in \ker\left(X_\alpha{}^\beta\right)$$

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$$h^\sigma_\alpha\left(\mathbf{p}, s, \tilde{s}\right) = D(\tilde{\Lambda})_\alpha{}^\beta \; h^\sigma_\alpha\left(s, \tilde{s}\right)$$

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$$h_\alpha^\sigma\left(\mathbf{p}, s, \tilde{s}\right) = D(\tilde{\Lambda})_\alpha{}^\beta\ h_\alpha^\sigma\left(s, \tilde{s}\right)$$

$\ker\left(X_\alpha{}^\beta\right) \subsetneq [\alpha]$ **Gauge transformation is needed**

$\ker\left(X_\alpha{}^\beta\right) = [\alpha]$ **Gauge invariance**

Amplitudes with massless particle

- **Partial wave formula**

$$h_\alpha^\sigma(\mathbf{p}, s, \tilde{s}) = \underbrace{D_\alpha{}^\beta(\Lambda)}_{\text{pure-orbital part}} \times \underbrace{u_\beta^{\sigma'}(s)\ D_{\sigma'}^{(s)\sigma}(R)}_{\text{pure-spin part}}$$

Amplitudes with massless particle

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$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1, \mathbf{p}_2^*, \mathbf{p}_3^*; L, S) = \Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1, \mathbf{p}_2^*, \mathbf{p}_3^*; L, S)\; \bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2'}(s_2)\, u_{\alpha_3}^{\sigma_3'}(s_3)\; D_{\sigma_2'}^{(s_2)\sigma_2}(R_{2^*})\; D_{\sigma_3'}^{(s_3)\sigma_3}(R_{3^*})$$

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- **From c.m.f. to any frame**

Amplitudes with massless particle

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- **From c.m.f. to any frame**

$$D_\alpha{}^\beta(\Lambda_1)\;\, h_\beta^\sigma(\mathbf{p}_i^*, s_i, \tilde{s}_i) \simeq h_\alpha^{\sigma'}(\mathbf{p}_i, s_i, \tilde{s}_i)\;\, D_{\sigma'}^{(s_i)\sigma}(\tilde{R}_{1i}) \;\left[\tilde{R}_{1i} = R_i^{-1} \cdot R_{1i} \cdot R_{i^*}\right]\; (i = 2, 3)$$

Amplitudes with massless particle

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- **Weight function for linear independent (L,S) bases**

Amplitudes with massless particle

- **Partial wave formula**

$$h_\alpha^\sigma(\mathbf{p}, s, \tilde{s}) = \underbrace{D_\alpha{}^\beta(\Lambda)}_{\text{pure-orbital part}} \times \underbrace{u_\beta^{\sigma'}(s)\; D_{\sigma'}^{(s)\sigma}(R)}_{\text{pure-spin part}}$$

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{k}_1, \mathbf{p}_2^*, \mathbf{p}_3^*; L, S) = \Gamma_{\alpha_1}^{\alpha_2\alpha_3}(\mathbf{k}_1, \mathbf{p}_2^*, \mathbf{p}_3^*; L, S)\, \bar{u}_{\sigma_1}^{\alpha_1}(s_1)\, u_{\alpha_2}^{\sigma_2'}(s_2)\, u_{\alpha_3}^{\sigma_3'}(s_3)\, D_{\sigma_2'}^{(s_2)\sigma_2}(R_{2^*})\, D_{\sigma_3'}^{(s_3)\sigma_3}(R_{3^*})$$

- **From c.m.f. to any frame**

$$D_\alpha{}^\beta(\Lambda_1)\; h_\beta^\sigma(\mathbf{p}_i^*, s_i, \tilde{s}_i) \simeq h_\alpha^{\sigma'}(\mathbf{p}_i, s_i, \tilde{s}_i)\; D_{\sigma'}^{(s_i)\sigma}(\tilde{R}_{1i}) \;\left[\tilde{R}_{1i} = R_i^{-1}\cdot R_{1i}\cdot R_{i^*}\right]\; (i = 2, 3)$$

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3; L, S) = \mathcal{A}_{\sigma_1}^{\sigma_2'\sigma_3'}(\mathbf{k}_1, \mathbf{p}_2^*, \mathbf{p}_3^*; L, S)\; D_{\sigma_2'}^{(s_2)\sigma_2}\left(\tilde{R}_{12}^{-1}\right)\; D_{\sigma_3'}^{(s_2)\sigma_3}\left(\tilde{R}_{13}^{-1}\right)$$

- **Weight function for linear independent (L,S) bases**

Two or three massless particles: $W(s_1, s_2, s_3, L, S) = F_S(s_1, s_2, s_3, S) + F_L(s_1, s_2, s_3, L, S) + F_\sigma(s_1, s_2, s_3, L, S)$

One massless particle: $W(s_1, s_2, s_3, L, S) = F_S(s_1, s_2, s_3, S)$

$$\begin{aligned} F_S(s_1, s_2, s_3, S) &= -(s_2+s_3+1)\,|S-s_1| + S \\ F_L(s_1, s_2, s_3, L, S) &= -2(s_2+s_3+1)^2\left|L - |S-s_1| - \frac{1}{2}\right| \end{aligned}$$

$$F_\sigma(s_1, s_2, s_3, L, S) = \begin{cases} -2(s_2+s_3+1)^2(s_1+s_2+s_3) & \text{for } \left(C_{s_1}^{LS}\right)_{s_2\pm s_3}^{0\, s_2\pm s_3} = 0 \\ 0 & \text{for others} \end{cases}$$

Numerical calculation part

- **Partial wave formula for numerical calculation**

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}\left(\mathbf{k}_1, \mathbf{p}_2^*, \mathbf{p}_3^*; L, S\right) = \frac{|\mathbf{p}_2^*|^L}{\sqrt{2s_1+1}} \left(C_{s_1}^{SL}\right)_{\sigma_1}^{(\sigma_2+\sigma_3)\sigma_L} \left(C_S^{s_2 s_3}\right)_{\sigma_2+\sigma_3}^{\sigma_2\sigma_3} Y_{L,\sigma_L}\left(\hat{\mathbf{p}}_{2^*}\right)$$

Numerical calculation part

- **Partial wave formula for numerical calculation**

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3}\left(\mathbf{k}_1, \mathbf{p}_2^*, \mathbf{p}_3^*; L, S\right) = \frac{|\mathbf{p}_2^*|^L}{\sqrt{2s_1+1}} \left(C_{s_1}^{SL}\right)_{\sigma_1}^{(\sigma_2+\sigma_3)\sigma_L} \left(C_S^{s_2 s_3}\right)_{\sigma_2+\sigma_3}^{\sigma_2\sigma_3} Y_{L,\sigma_L}\left(\hat{\mathbf{p}}_{2^*}\right)$$

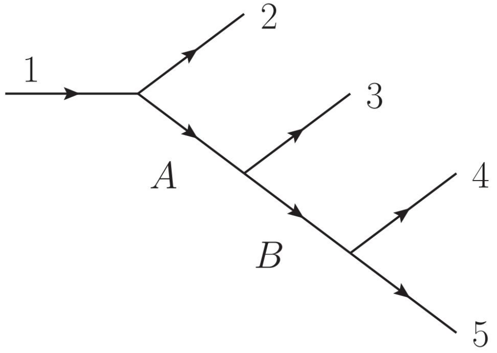
- **Transformation Wigner-D**

```
MassiveTrans[{pt_, px_, py_, pz_}, {qt_, qx_, qy_, qz_}, s_] :=
  Module[{p, q, hat, m1, m2, γ1, γ2, cos, nhat, θ, ϕ, ψ, res},
  模块
   p = √(px^2 + py^2 + pz^2);
   q = √(qx^2 + qy^2 + qz^2);
   hat = {py qz - pz qy, pz qx - px qz, px qy - py qx};
   If[Or[p q == 0, hat == {0, 0, 0}], res = IdentityMatrix[2 s + 1],
   … 逻辑或                              单位矩阵
    m1 = √(pt^2 - p^2); γ1 = pt / m1;
    m2 = √(qt^2 - q^2); γ2 = qt / m2;
    cos = (px qx + py qy + pz qz) / (-p q); nhat = hat / (p q √(1 - cos^2));
    {θ, ϕ} = If[Or[nhat[[1]] ≠ 0, nhat[[2]] ≠ 0],
                 … 逻辑或
      {ArcCos[nhat[[3]]], If[nhat[[2]] ≥ 0, ArcCos[nhat[[1]] / (√(1 - nhat[[3]]^2))],
       反余弦              如果                反余弦
        2 π - ArcCos[nhat[[1]] / (√(1 - nhat[[3]]^2))]]}, {0, 0}];
              反余弦
    ψ = ArcCos[1 - ((γ1 - 1) (γ2 - 1) (1 - cos^2)) / (1 + γ1 γ2 + (p / m1) (q / m2) cos)];
        反余弦
    res = DFunc[s, -ϕ, -θ, -ψ].DFunc[s, 0, θ, ϕ];
   ];
   res
  ];
```

```
MasslessTrans[{pt_, px_, py_, pz_}, {qt_, qx_, qy_, qz_}, s_] :=
  Module[{p, hat, M, λ, qs1, qs2, qs3, pq, Δ, x1, x2, x3, δ1, δ2, δ3, cos, sign, ψ, res},
  模块
   If[Or[qx ≠ 0, qy ≠ 0], p = √(px^2 + py^2 + pz^2);
   … 逻辑或
    hat = {py qz - pz qy, pz qx - px qz, px qy - py qx};
    M = √(pt^2 - p^2); pq = pt qt - px qx - py qy - pz qz; λ = (M qt + pq) / (M (M + pt));
    qs3 = (λ pz - qz); Δ = (pq^2 - M^2 qs3^2);
    If[Or[p == 0, hat == {0, 0, 0}, Δ == 0], res = IdentityMatrix[2 s + 1],
    … 逻辑或                                      单位矩阵
     qs1 = (λ px - qx); qs2 = (λ py - qy);
     x1 = p^2 px (M^2 py qs2 qs3 - pz Δ);
     x2 = p^2 py (M^2 px qs1 qs3 - pz Δ);
     x3 = M^3 p^2 pz (px qs1 + py qs2) qs3;
     δ1 = (M p^2 pt + pt^2 px^2 + M^2 (py^2 + pz^2)) M^2 qs1 qs3 + x1;
     δ2 = (pt py^2 + M (px^2 + pz^2)) M^2 (M + pt) qs2 qs3 + x2;
     δ3 = x3 - (M (px^2 + py^2) + pt pz^2) M (M + pt) Δ;
     cos = (M qx qz δ1 + M qy qz δ2 - (qt^2 - qz^2) δ3) / (M^2 p^2 (M + pt) pq qt √((qt^2 - qz^2) Δ));
     sign = (-qy δ1 + qx δ2);
     ψ = If[sign ≥ 0, ArCcos[cos], 2 π - ArcCos[cos]]; res = DFunc[s, 0, 0, ψ]];,
         如果         反余弦              反余弦
    res = DFunc[s, 0, 0, (-π) / 2];];
   res
  ];
```

Numerical calculation part

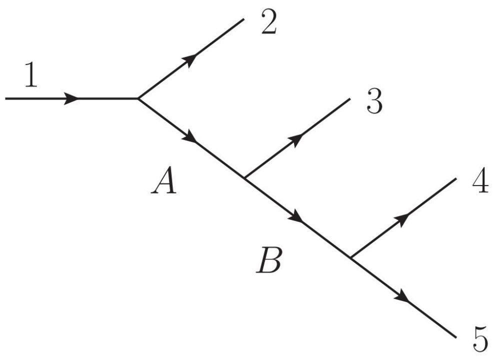
- **Combine with isobar model**



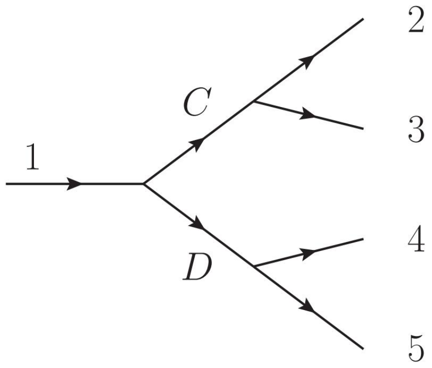
$$(\mathcal{A}_1)_{\sigma_1}^{\sigma_2\sigma_3\sigma_4\sigma_5} \;=\; \mathcal{A}_{\sigma_1}^{\sigma_A\sigma_2}\;\mathcal{A}_{\sigma_A}^{\sigma_B\sigma_3}\;\mathcal{A}_{\sigma_B}^{\sigma_4\sigma_5} \;\times\; \mathrm{BW}_1\;\mathrm{FF}_1$$

Numerical calculation part

- **Combine with isobar model**



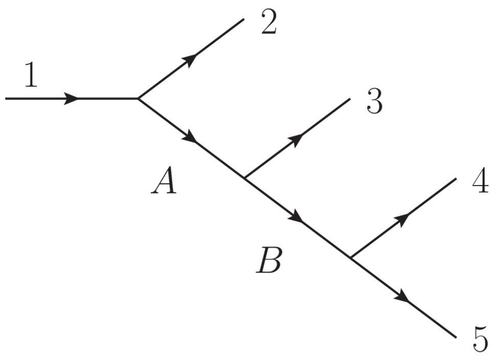
$$(\mathcal{A}_1)_{\sigma_1}^{\sigma_2\sigma_3\sigma_4\sigma_5} = \mathcal{A}_{\sigma_1}^{\sigma_A\sigma_2}\ \mathcal{A}_{\sigma_A}^{\sigma_B\sigma_3}\ \mathcal{A}_{\sigma_B}^{\sigma_4\sigma_5} \times \mathrm{BW}_1\ \mathrm{FF}_1$$

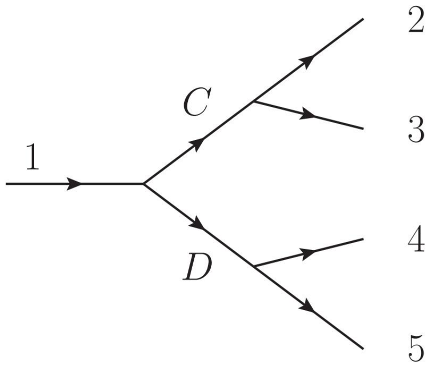


$$(\mathcal{A}_2)_{\sigma_1}^{\sigma_2\sigma_3\sigma_4\sigma_5} = \mathcal{A}_{\sigma_1}^{\sigma_C\sigma_D}\ \mathcal{A}_{\sigma_C}^{\sigma_2\sigma_3}\ \mathcal{A}_{\sigma_D}^{\sigma_4\sigma_5} \times \mathrm{BW}_2\ \mathrm{FF}_2$$

Numerical calculation part

- **Combine with isobar model**



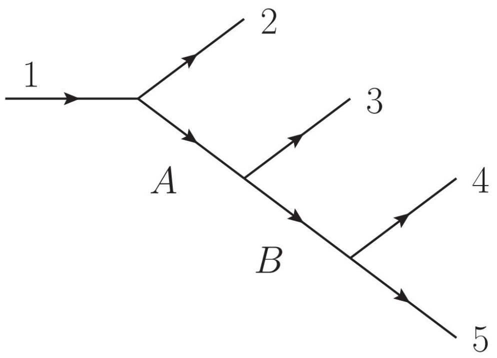


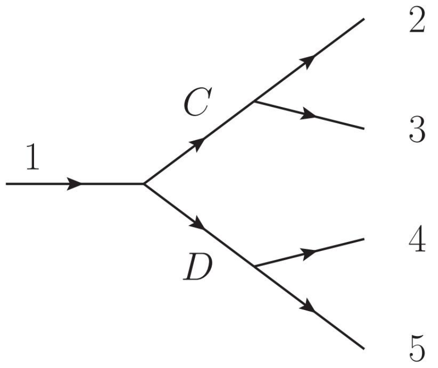
$$(\mathcal{A}_1)_{\sigma_1}^{\sigma_2\sigma_3\sigma_4\sigma_5} = \mathcal{A}_{\sigma_1}^{\sigma_A\sigma_2}\ \mathcal{A}_{\sigma_A}^{\sigma_B\sigma_3}\ \mathcal{A}_{\sigma_B}^{\sigma_4\sigma_5} \times \mathrm{BW}_1\ \mathrm{FF}_1 \qquad (\mathcal{A}_2)_{\sigma_1}^{\sigma_2\sigma_3\sigma_4\sigma_5} = \mathcal{A}_{\sigma_1}^{\sigma_C\sigma_D}\ \mathcal{A}_{\sigma_C}^{\sigma_2\sigma_3}\ \mathcal{A}_{\sigma_D}^{\sigma_4\sigma_5} \times \mathrm{BW}_2\ \mathrm{FF}_2$$

$$(\mathcal{A}_{\mathrm{tot}})_{\sigma_1}^{\sigma_2\sigma_3\sigma_4\sigma_5} = (\mathcal{A}_1)_{\sigma_1}^{\sigma_2\sigma_3\sigma_4\sigma_5} + (\mathcal{A}_2)_{\sigma_1}^{\sigma_2\sigma_3\sigma_4\sigma_5}$$

Numerical calculation part

- **Combine with isobar model**





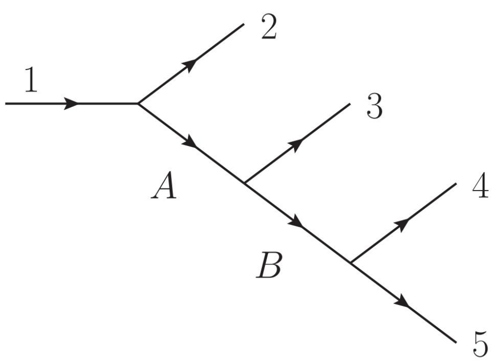
$$(\mathcal{A}_1)_{\sigma_1}^{\sigma_2\sigma_3\sigma_4\sigma_5} = \mathcal{A}_{\sigma_1}^{\sigma_A\sigma_2}\ \mathcal{A}_{\sigma_A}^{\sigma_B\sigma_3}\ \mathcal{A}_{\sigma_B}^{\sigma_4\sigma_5} \times \mathrm{BW}_1\ \mathrm{FF}_1 \qquad (\mathcal{A}_2)_{\sigma_1}^{\sigma_2\sigma_3\sigma_4\sigma_5} = \mathcal{A}_{\sigma_1}^{\sigma_C\sigma_D}\ \mathcal{A}_{\sigma_C}^{\sigma_2\sigma_3}\ \mathcal{A}_{\sigma_D}^{\sigma_4\sigma_5} \times \mathrm{BW}_2\ \mathrm{FF}_2$$

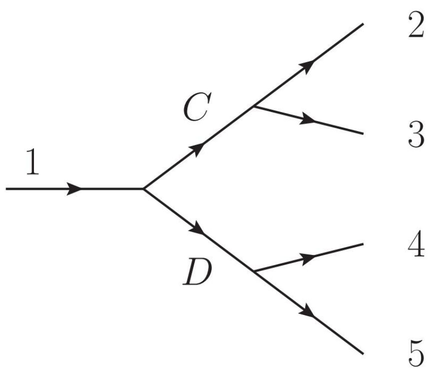
$$(\mathcal{A}_{\mathrm{tot}})_{\sigma_1}^{\sigma_2\sigma_3\sigma_4\sigma_5} = (\mathcal{A}_1)_{\sigma_1}^{\sigma_2\sigma_3\sigma_4\sigma_5} + (\mathcal{A}_2)_{\sigma_1}^{\sigma_2\sigma_3\sigma_4\sigma_5}$$

- **Crosscheck this scheme with the helicity amplitude (TFPWA)**
  [Collaborate with Yi Jiang, Run-Qiu Ma and Shi Wang]
  1. Boson case
  2. Fermion case
  3. Radiative decay

Numerical calculation part

- **Combine with isobar model**





$$(\mathcal{A}_1)^{\sigma_2\sigma_3\sigma_4\sigma_5}_{\sigma_1} = \mathcal{A}^{\sigma_A\sigma_2}_{\sigma_1}\,\mathcal{A}^{\sigma_B\sigma_3}_{\sigma_A}\,\mathcal{A}^{\sigma_4\sigma_5}_{\sigma_B} \times \mathrm{BW}_1\,\mathrm{FF}_1 \qquad (\mathcal{A}_2)^{\sigma_2\sigma_3\sigma_4\sigma_5}_{\sigma_1} = \mathcal{A}^{\sigma_C\sigma_D}_{\sigma_1}\,\mathcal{A}^{\sigma_2\sigma_3}_{\sigma_C}\,\mathcal{A}^{\sigma_4\sigma_5}_{\sigma_D} \times \mathrm{BW}_2\,\mathrm{FF}_2$$

$$(\mathcal{A}_{\mathrm{tot}})^{\sigma_2\sigma_3\sigma_4\sigma_5}_{\sigma_1} = (\mathcal{A}_1)^{\sigma_2\sigma_3\sigma_4\sigma_5}_{\sigma_1} + (\mathcal{A}_2)^{\sigma_2\sigma_3\sigma_4\sigma_5}_{\sigma_1}$$

- **Crosscheck this scheme with the helicity amplitude (TFPWA)**
  [Collaborate with Yi Jiang, Run-Qiu Ma and Shi Wang]
  1. **Boson case** ??
  2. **Fermion case** ✓
  3. **Radiative decay** ??

Summary

- General framework of covariant LS scheme
  - Unique building block: spin wave function
  - Both the helicity amplitude and the covariant tensor amplitude can be constructed through this framework
  - Suitable for both massive and massless particles
- Numerical calculation programs can be easily implemented

Thank you for your attention!

Back up

$$u^{\sigma}_{\alpha_1\alpha_2}(\chi, s) = \sum_{s_1,s_2} \sqrt{(2s_1+1)(2s_2+1)(2s_L+1)(s_R+1)} \begin{Bmatrix} s_{1L} & s_{1R} & s_1 \\ s_{2L} & s_{2R} & s_2 \\ s_L & s_R & s \end{Bmatrix} \times \left(C^{s}_{s_1 s_2}\right)^{\sigma}_{\sigma_1\sigma_2} u^{\sigma_1}_{\alpha_1}(s_1)\, u^{\sigma_2}_{\alpha_2}(s_2)$$

$$u^{\sigma}_{\alpha_1\alpha_2}(\mathbf{p}, \chi, s) \equiv D_{\alpha_1}{}^{\beta_1}(\Lambda)\, D_{\alpha_2}{}^{\beta_2}(\Lambda)\, u^{\sigma}_{\beta_1\beta_2}(\chi, s)$$

$$\bar{u}^{\alpha_1\alpha_2}_{\sigma}(\chi^*, s) = \sum_{s_1,s_2} \sqrt{(2s_1+1)(2s_2+1)(2s_L+1)(s_R+1)} \begin{Bmatrix} s_{1L} & s_{1R} & s_1 \\ s_{2L} & s_{2R} & s_2 \\ s_L & s_R & s \end{Bmatrix} \times \left(C^{s_1 s_2}_{s}\right)^{\sigma_1\sigma_2}_{\sigma} \bar{u}^{\alpha_1}_{\sigma_1}(s_1)\, \bar{u}^{\alpha_2}_{\sigma_2}(s_2)$$

$$[\alpha] = [\mu] \qquad X = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & 0 & 0 \end{pmatrix} \qquad \ker X = \{\varepsilon^{\sigma}_{\mu} | \sigma = \pm 1\}$$

$$[\alpha] = [\mu, \nu] \qquad X = 0$$

$$\ker X = [\alpha]$$

<table>
<tr><th>Range</th><th>$N_0(s_1; s_2, s_3)$</th><th>$N_1(s_1; s_2, s_3)$</th><th>$N_2(s_1; s_2, s_3)$</th><th>$N_3(s_1; s_2, s_3)$</th></tr>
<tr><td>(a)</td><td rowspan="2">$(2s_1 + 1)(2s_3 + 1)$</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td>(b)</td><td rowspan="2">$2(s_1 - s_2 + s_3 + 1)$</td><td rowspan="3">2</td><td>2</td></tr>
<tr><td>(c)</td><td>$n(s_1; s_2, s_3)$</td><td>0</td></tr>
<tr><td>(d)</td><td rowspan="2">$(2s_1 + 1)(2s_2 + 1)$</td><td rowspan="2">$2(2s_1 + 1)$</td><td>2</td></tr>
<tr><td>(e)</td><td>0</td><td>0</td></tr>
<tr><td>(f)</td><td rowspan="2">$(2s_2 + 1)(2s_3 + 1)$</td><td rowspan="2">$2(2s_3 + 1)$</td><td rowspan="2">4</td><td>2</td></tr>
<tr><td>(g)</td><td>0</td></tr>
</table>

Table 4. The number of linearly independent terms of three-particle amplitudes $(s_1 \to s_2 + s_3)$ is determined in seven different ranges: (a) $s_1 < s_2 - s_3$; (b) $s_1 = s_2 - s_3$; (c) $|s_2 - s_3| < s_1 < s_2 + s_3$; (d) $s_1 = s_3 - s_2$; (e) $s_1 < s_3 - s_2$; (f) $s_1 = s_2 + s_3$; (g) $s_1 > s_2 + s_3$. $N_i(s_1; s_2, s_3)$ $(i = 0, 1, 2, 3)$ represents the number of linearly independent terms for four cases: $(i = 0)$ all particles are massive; $(i = 1)$ particle-2 is massless and $s_2 \neq 0$; $(i = 2)$ both particle-2 and 3 are massless and $s_{2,3} \neq 0$; $(i = 3)$ all particles are massless and $s_{1,2,3} \neq 0$. The value of $n(s_1; s_2, s_3)$ is calculated as follows: $n(s_1; s_2, s_3) = -(s_1^2 + s_2^2 + s_3^2) + 2(s_1 s_2 + s_2 s_3 + s_1 s_3) + s_1 + s_2 + s_3 + 1$.

- E.g. consider $J/\psi \to \gamma f_2$.
  - the possible (S, L) combinations are $(1,0), (1,2), (2,2), (3,2), (3,4)$.
  - based on the calculation in the previous slide, one has (in helicity basis),

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3} \propto \left[\begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 \end{pmatrix}\right]_{\sigma_1}^{\sigma_2\sigma_3}$$

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3} \propto \left[\begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 \end{pmatrix}\right]_{\sigma_1}^{\sigma_2\sigma_3}$$

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3} \propto \left[\begin{pmatrix} 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{\frac{3}{2}} & 0 & 0 \end{pmatrix}\right]_{\sigma_1}^{\sigma_2\sigma_3}$$

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3} \propto \left[\begin{pmatrix} 0 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -\frac{3}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \end{pmatrix}\right]_{\sigma_1}^{\sigma_2\sigma_3}$$

$$\mathcal{A}_{\sigma_1}^{\sigma_2\sigma_3} \propto \left[\begin{pmatrix} 0 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -2\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2\sqrt{2} & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \end{pmatrix}\right]_{\sigma_1}^{\sigma_2\sigma_3}$$

  - only 3 of the 5 amplitudes are linearly independent !

```
s1 = 1; s2 = 1; s3 = 2;
WFunc1[s1_, s2_, s3_, S_, L_] :=
  - (s2 + s3 + 1) Abs[S - s1] + S;

WFunc1[s1, s2, s3, 1, 0]
WFunc1[s1, s2, s3, 1, 2]
WFunc1[s1, s2, s3, 2, 2]
WFunc1[s1, s2, s3, 3, 2]
WFunc1[s1, s2, s3, 3, 4]

1

1

-2

-5

-5
```