Thermal Equilibrium and First Law of Thermodynamics for a Charged Star in Gravitational and Electromagnetic Fields

$$\delta E = \tilde{T}\,\delta S + \tilde{\mu}\,\delta N + \Omega\,\delta J$$

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Outline

Motivations

Charged perfect fluid

fixed background

Dynamical background

Outlooks

Motivations

High Energy Community

Minwalla, Kovtun, Loganayagam, Liu ...

fixed background.

no rotation.

no derivation.

General Relativity Community

Katz, Manor, Wald, Gao ...

dynamical background.

no electromagnetic field.

not on the equal footing.

Ordinary first law of thermodynamics

$$dE = TdS - pdV + \mu dN$$

$$E(\lambda S, \lambda V, \lambda N) = \lambda E(S, V, N)$$

$$E = TS - pV + \mu N$$

$$\underline{\rho + p = Ts + \mu n}$$

$$\underline{d\rho} = d\left(\frac{E}{V}\right) = \frac{dE}{V} - \frac{E}{V^2} dV$$

$$= \frac{TdS - pdV + \mu dN}{V} - \frac{TS - pV + \mu N}{V^2} dV$$

$$= Td\left(\frac{S}{V}\right) + \mu d\left(\frac{N}{V}\right) = \underline{Tds + \mu dn}$$

$$\underline{dp = sdT + nd\mu}$$

Charged perfect fluid

$$T_{ab} = (\rho + p) u_a u_b + p g_{ab}$$

$$J_a = e n u_a$$

$$u_a u^a = g_{ab} u^a u^b = -1$$

$\nabla_a J^a = 0$ the conservation of charge and particle number

$$\nabla_a T^{ab} = F^{bc} J_c \qquad F = dA$$

$$0 = u_b \nabla_a T^{ab} = -u^a \nabla_a (\rho + p) - (\rho + p) \nabla_a u^a$$

$$+ (\rho + p) u^a u_b \nabla_a u^b + u^a \nabla_a p =$$

$$-\left(u^a \nabla_a \rho + (\rho + p) \nabla_a u^a\right) =$$

$$-\left(u^a (\nabla_a s T + \nabla_a n \mu) + (Ts + \mu n) \nabla_a u^a\right)$$

$$= -T \nabla_a (s u^a) - \mu \nabla_a (n u^a)$$

$$\underline{= -T \nabla_a (s u^a)}$$ the conservation of entropy

Thermodynamics in fixed background

ξ is a vector field, satisfying

$$\mathcal{L}_\xi g_{ab} = \mathcal{L}_\xi A_a = 0$$

$$\nabla_a \left(T^{ab}\xi_b\right) = \nabla_a T^{ab}\xi_b = F^{bc}\xi_b J_c$$

$$= (\xi \cdot dA)_c J^c = \left[\mathcal{L}_\xi A_c - d_c(\xi \cdot A)\right] J^c$$

$$= -\nabla_c\left(\xi^b A_b\right) J^c = -\nabla_a\left(J^a A^b \xi_b\right)$$

$$j_\xi^a = \left(T^{ab} + J^a A^b\right)\xi_b$$

$\nabla_a j_\xi^a = 0$ Noether theorem associated

with ξ^a

t^a timelike φ^a axial

$$u^a = \frac{t^a + \Omega \varphi^a}{|V|}$$ Circular flow

$$|V|^2 = -g_{ab}(t^a + \Omega \varphi^a)(t^b + \Omega \varphi^b)$$

$$E = -\int_\Sigma (T^{ab} + J^a A^b) t_b \, \epsilon_{adef}$$

$$j = \int_\Sigma (T^{ab} + J^a A^b) \varphi_b \, \epsilon_{adef}$$

$$\delta E = -\int_\Sigma (\delta T^{ab} + \delta J^a A^b) t_b \, \epsilon_{adef}$$

$$= -\int_\Sigma (\delta T^{ab} + \delta J^a A^b)(|V| u_b - \Omega \varphi_b) \epsilon_{adef}$$

$$= \int_\Sigma \Omega \, \delta \underline{j} - \int_\Sigma |V| (\delta T^{ab} + \delta J^a A^b) u_b \, \epsilon_{adef}$$

$$\delta T^{ab} u_b = \Big[\delta(\rho+p)\, u^a u^b + (\rho+p)\left(\delta u^a u^b + u^a \delta u^b\right)$$

$$+ \delta p\, g^{ab}\Big] u_b = -\delta(\rho+p)\, u^a$$

$$-(\rho+p)\,\delta u^a + \delta p\, u^a$$

$$= -\delta\rho\, u^a - (\rho+p)\,\delta u^a$$

$$= -\Big[\left(T\delta s + \mu\,\delta n\right) u^a + \left(Ts + \mu n\right)\delta u^a\Big]$$

$$= -\left[T\delta(s u^a) + \mu\,\delta(n u^a)\right]$$

$$\delta J^a A^b u_b = \delta(n u^a)\left(A^b u_b e\right)$$

$$0 = \delta\left(g_{ab} u^a u^b\right) = g_{ab}\,\delta u^a u^b + g_{ab} u^a \delta u^b$$

$$= 2\,\delta u^b u_b$$

$$\delta E = \int_{\Sigma} \left(\Omega\, \delta \underline{j} + \tilde{T}\, \delta \underline{S} + \tilde{\mu}\, \delta \underline{N} \right)$$

$$\tilde{T} = |V|\, T \qquad \tilde{\mu} = |V| (\mu - e u^b A_b)$$

Thermal equilibrium:

$\delta S = 0$ when $\delta E = \delta N = \delta j = 0$

$\Omega, \tilde{T}, \tilde{\mu}$ are uniform $\Rightarrow$

Thermal equilibrium

Thermal equilibrium $\Rightarrow \exists\ \delta j_1 = \delta N_1 = 0$

$\delta S_1 \neq 0 \quad \delta E_1 \neq 0$

Suppose that $\tilde{T}$ is not uniform, then

$\delta \underline{j}_2 = \delta \underline{N}_2 = 0 \quad \delta \underline{S}_2 = \alpha (\tilde{T} - \bar{\tilde{T}}) \nu^a \epsilon_{abcd}$

such that

$$\delta S_2 = \alpha \int (\tilde{T} - \bar{\tilde{T}}) v^a \epsilon_{abcd} = 0$$

$$\delta E_2 = \alpha \int_\Sigma \tilde{T} (\tilde{T} - \bar{\tilde{T}}) v^a \epsilon_{abcd}$$

$$= \alpha \int_\Sigma (\tilde{T} - \bar{\tilde{T}})^2 v^a \epsilon_{abcd} = \delta E_1$$

By combining such two perturbations we have $\delta E = \delta N = \delta \hat{j} = 0$ but $\delta S \neq 0$. which contradicts with the fact that the fluid is in thermal equilibrium. Therefore $\tilde{T}$ must be uniform.

$$\delta E = \Omega \delta \dot{j} + \tilde{T} \delta S + \tilde{\mu} \delta N$$

thermodynamics in dynamical background

$$\mathcal{U}_{ab} = E_{ab} - \frac{1}{2} T_{ab} = 0$$

$$\mathcal{U}^a = E^a + J^a = 0$$

$$E_{ab} = G_{ab} - \frac{1}{2} T_{ab}^{em}$$

$$T_{ab}^{EM} = F_{ac} F_b{}^c - \frac{1}{4} F_{cd} F^{cd} g_{ab}$$

$$E^a = D_b F^{ba}$$

Iyer–Wald formalism

$$\underline{L} = \left(R - \frac{1}{4} F_{ab} F^{ab}\right) \underline{\epsilon}$$

$$\delta \underline{L} = (E_{ab}\, \delta g^{ab} + E^a \delta A_a)\, \underline{\epsilon} + d\, \underline{\Theta}$$

The presymplectic potential

$$\underline{\Theta}(\phi, \delta\phi) = \underline{\Theta}_{GR} + \underline{\Theta}_{EM}$$

$$= w_{GR} \cdot \underline{\epsilon} + w_{EM} \cdot \underline{\epsilon}$$

$$w^a_{GR} = g^{ab} g^{cd} (\nabla_d \delta g_{bc} - \nabla_b \delta g_{cd})$$

$$w^a_{EM} = - F^{ab} \delta A_b$$

The Noether current associated with a vector field χ

$$\underline{J}_\chi \equiv \underline{\Theta}_{GR}(\mathcal{L}_\chi g) + \underline{\Theta}_{EM}(\mathcal{L}_\chi A) - \chi \cdot \underline{L}$$

$$= d\underline{Q}_\chi + C_\chi \cdot \underline{\epsilon}$$

$$\underline{Q}_\chi = Q_\chi^{GR} + Q_\chi^{EM} = - * d\chi - * F A_b \chi^b$$

$$C_\chi^a = (2E^a{}_b - E^a A_b)\chi^b$$

$$\delta \underline{\Theta}(\phi, \mathcal{L}_\chi \phi) - \chi \cdot \delta \underline{L}$$
$$= d\,\delta \underline{Q}_\chi + \delta(C_\chi \cdot \underline{\epsilon})$$

$$\delta \underline{\Theta}(\phi, \mathcal{L}_\chi \phi) - \chi \cdot \underline{\epsilon}(E_{ab}\,\delta g^{ab} + E^a \delta A_a)$$
$$- \chi \cdot d\,\underline{\Theta} = d\,\delta \underline{Q}_\chi + \delta(C_\chi \cdot \underline{\epsilon})$$

$$\delta \underline{\Theta}(\phi, \mathcal{L}_\chi \phi) - (\mathcal{L}_\chi \underline{\Theta} - d(\chi \cdot \underline{\Theta}))$$
$$- \chi \cdot \underline{\epsilon}\,(E_{ab}\,\delta g^{ab} + E^a \delta A_a)$$
$$= d\,\delta Q_\chi + \delta(C_\chi \cdot \underline{\epsilon})$$

$$d(\delta Q_\chi - \chi \cdot \underline{\Theta}) = \omega(\phi, \delta\phi, \mathcal{L}_\chi \phi)$$
$$- \chi \cdot \underline{\epsilon}(E_{ab}\,\delta g^{ab} + E^a \delta A_a) - \delta(C_\chi \cdot \underline{\epsilon})$$

$$\omega(\phi, \delta_1\phi, \delta_2\phi) = \delta_1 \underline{\Theta}(\phi, \delta_2\phi) - \delta_2 \underline{\Theta}(\phi, \delta_1\phi)$$

$$d(\delta Q_\xi - \xi\cdot\underline{\Theta}) = -\xi\cdot\underline{\epsilon}\left(E_{ab}\,\delta g^{ab} + E^a\,\delta A_a\right) - \delta\left(C_\xi\cdot\underline{\epsilon}\right)$$

$$\delta M \equiv \delta\int_{S_\infty} \underset{\sim}{Q}_t - t\cdot\underline{B} = \int_{S_\infty} \delta Q_t - t\cdot\underline{\Theta}$$

$$= \int_\Sigma \left[-t\cdot\underline{\epsilon}\left(E_{ab}\,\delta g^{ab} + E^a\,\delta A_a\right) - \delta\left(C_t\cdot\underline{\epsilon}\right)\right]$$

$$\delta j = -\delta\int_{S_\infty} \underset{\sim}{Q}_\varphi = \int_\Sigma \delta\left(C_\varphi\cdot\epsilon\right)$$

$$\delta M = \int_\Sigma -|v|\left(E_{ab}\,\delta g^{ab} + E^a\,\delta A_a\right)u^c\,\epsilon_{cdef}$$

$$-|v|\,u^b\,\delta\left[\left(2E^c{}_b - E^c A_b\right)\epsilon_{cdef}\right]$$

$$+\,\Omega\,\delta \underline{j}$$

$$J^a \delta A_a u^c \epsilon_{cdef} - u^b \delta(J^c A_b \epsilon_{cdef}) = -e u^b A_b \delta \underline{N}$$

$$-\frac{1}{2} T_{ab} \delta g^{ab} u^c \epsilon_{cdef} - u^b \delta(T^c{}_b \epsilon_{cdef}) = u \delta \underline{N} + T \delta \underline{S}$$

$$\delta M = \int_{\underline{S}} \tilde{T} \delta \underline{S} + \tilde{u} \delta \underline{N} + \Omega \delta \underline{j}$$

$$-\int |N| (H_{ab} \delta g^{ab} + H^a \delta A_a) u^c \epsilon_{cdef}$$

$$-\int |N| u^b \delta \left((2H^c{}_b - H^c A_b) \epsilon_{cdef} \right)$$

$$= \int_{\Sigma} \tilde{T} \delta \underline{S} + \tilde{u} \delta \underline{N} + \Omega \delta \underline{j}$$

Different from the case of fixed backgrounds $\delta \underline{S}$, $\delta \underline{N}$, $\delta \underline{j}$ are subject to the constraint equations. But it turns out that they can still be chosen freely!

Outlooks

High dimensions

Other boundaries

Finite region, AdS

High derivative theories

Gauss-Bonnet, Dirac-Born-Infeld

Thermal Equilibrium → Dynamic one

Stability $\delta^2 S \leq 0$

The constancy of Ω

Thanks for your listening !!!