RUHR UNIVERSITÄT BOCHUM RUB

Two-particle scattering in the finite volume using plane wave basis

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Based on arXiv: 2108.02709

Together with Evgeny Epelbaum (RUB)

Contents

Introduction

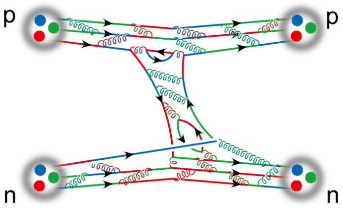
lattice QCD and finite volume energy levels

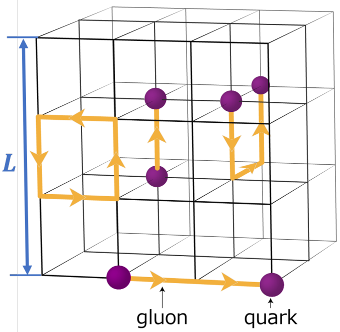
- QCD is the fundamental theory of the strong interaction

$$\mathcal{L}_{QCD} = \sum_f \bar{q}_f(i\not{D} - \mathcal{M}q_f) - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu,a} \quad (1)$$

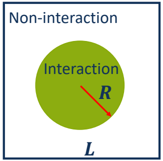
- Extract hadronic interaction (e.g. nuclear forces) from QCD?
- Lattice QCD: formulated on a lattice of points in space and time in a finite volume (FV)
- Lattice QCD results: energy levels in FV
- How to get the physical information from energy levels?
- Lüscher's formula: $E^{FV} \sim \delta^l$

Luscher:1990ux





Lüscher's formula and beyond



- Lüscher's formula (LF): **model-independent, one-to-one** $E^{FV} \sim \delta^l$
  - $\Rightarrow$ $L \gg R$, negligible $e^{-L/R}$ effect
  - $\Rightarrow$ Single channel, no partial wave (PW) mixture — Luscher:1990ux
- Long-range interaction: e.g. 1-π exchange for NN and $\bar{D}^* D/\bar{D}D^*[X(3872)]$ Sato:2007ms,Jansen:2015lha
- Partial wave mixing is unavoidable due to ~~rotational symmetry~~ in FV
  - $\Rightarrow$ ~~one-to-one~~, Parameterize T-matrix within theory, framework-dependent
- Alternative approaches: HAL QCD, UChPT in FV, Hamiltonian EFT... Ishii:2006ec,Doring:2011vk,Wu:2014vma,...
- IFV: partial wave expansion, reduce to 3D $\rightarrow$1D
- In FV: $|lm\rangle$ basis is not ideal?? ~~rotational symmetry~~; FV: discrete momentum
- Our work : **Plane Wave basis** expansion + effective field theory (EFT) Lee:2020fbo
  - $\Rightarrow$ Rest and moving systems, relativistic and non-relativistic systems

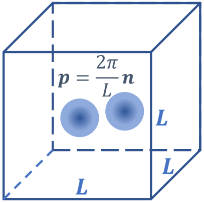
Lüscher's formula

Quantization of momentum

- Two frames: box frame (BF) and center of mass frame (CMF)
- boundary conditions in BF

$$\psi(\mathbf{x}_1, \mathbf{x}_2) = \psi(\mathbf{x}_1 + \mathbf{n}_1 L, \mathbf{x}_2 + \mathbf{n}_2 L) \tag{2}$$

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{P}, \quad \boldsymbol{p}_1 = \frac{2\pi}{L}\boldsymbol{n}, \quad \boldsymbol{P} = \frac{2\pi}{L}\boldsymbol{d}, \qquad \boldsymbol{n}, \boldsymbol{d} \in Z^3 \tag{3}$$



- $\boldsymbol{d} = (0,0,0)$: cubic group O_h
- The FV energy should be classified by irreps. of O_h group
- In order to obtain more E^{FV} in lattice QCD $\Rightarrow$moving systems in the box
- $\boldsymbol{d}^2 \neq 0$, quantization condition of ? $\boldsymbol{p}^* \equiv \boldsymbol{p}_1^*$ in CMF?

Rummukainen:1995vs,Leskovec:2012gb

Non-interacting systems

- Two particles on-shell, Lorentz boost to CMF

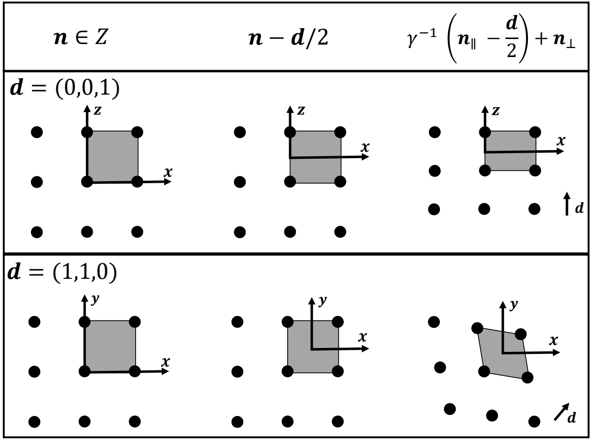
Rummukainen:1995vs,Leskovec:2012gb

$$\boldsymbol{p}_1^* = \gamma^{-1}\left(\boldsymbol{p}_{1\parallel} - \frac{1}{2}A\boldsymbol{P}\right) + \boldsymbol{p}_{1\perp}, \quad A \equiv 1 + \frac{m_1^2 - m_2^2}{E^{*2}}, \quad \gamma = \frac{\sqrt{E^{*2} + P^2}}{E^*} \tag{4}$$

- If $m_1 \neq m_2$, no space inversion symmetry

  Ziwen Fu, Phys.Rev.D85,014506; Leskovec:2012gb

- Focus on $m_1 = m_2$ with parity
- $\boldsymbol{d} = (0, 0, 1)$, D_{4h} group
- $\boldsymbol{d} = (1, 1, 0)$, D_{2h} group
- $\boldsymbol{d} = (1, 1, 1)$, D_{3d} group
- For non-relativistic system $\gamma = 1$



Interacting systems

- Scheme-I

Doring:2012eu

$$\boldsymbol{p}_1^* = \left[(\gamma-1)\frac{\boldsymbol{P}\cdot\boldsymbol{p}_1}{\boldsymbol{P}^2} - \frac{E_1}{E^*}\right]\boldsymbol{P} + \boldsymbol{p}_1 \tag{5}$$

$$E_1^* = \frac{EE_1 - \boldsymbol{P}\cdot\boldsymbol{p}_1}{E^*}, \quad \gamma = \frac{E}{E^*} = \frac{\sqrt{E^{*2}+P^2}}{E^*} \tag{6}$$

- Scheme-II: without extra E-dependence

Y.Li, J-J. Wu et al. Phys.Rev.D103,094518

$$\boldsymbol{p}_1^* = \left[(\gamma-1)\frac{\boldsymbol{P}\cdot\boldsymbol{p}_1}{\boldsymbol{P}^2} - \frac{\omega_1}{\sqrt{(\omega_1+\omega_2)^2 - P^2}}\right]\boldsymbol{P} + \boldsymbol{p}_1 \tag{7}$$

$$\omega_i = \sqrt{m_1^2 + \boldsymbol{p}_i^2}, \quad \gamma = \frac{\omega_1+\omega_2}{\sqrt{(\omega_1+\omega_2)^2 - \boldsymbol{P}^2}} \tag{8}$$

- The differences are exponentially suppressed
- For non-relativistic system, no ambiguity

Lüscher's formula

- Lippmann-Schwinger equation in FV: $T^L = K + KG_F T^L$ Luscher:1990ux,Polejaeva:2012ut

$$G_0^L(\boldsymbol{k}, z) = (\frac{2\pi}{L})^3 \sum_{\boldsymbol{p}} \frac{2\mu\delta^3(\boldsymbol{p}-\boldsymbol{k})}{\boldsymbol{p}^2 - q_0^2} = G_K(\boldsymbol{k}, z) + G_F(\boldsymbol{k}, z) \quad (9)$$

- Partial wave expansion: l is not a good quantum number

$$\langle \boldsymbol{p}|T^L|\boldsymbol{q}\rangle = 4\pi \sum_{l'm'lm} Y_{l'm'}(\hat{\boldsymbol{p}}) T^L_{l'm',lm}(p,q,z) Y^*_{lm}(\hat{\boldsymbol{q}})$$

$$\langle \boldsymbol{p}|K|\boldsymbol{q}\rangle = 4\pi \sum_{lm} Y_{lm}(\hat{\boldsymbol{p}}) K_l(p,q,z) Y^*_{lm}(\hat{\boldsymbol{q}}), \quad \tan\delta_l(q_0) = \frac{\mu p}{2\pi} K_l(q_0, q_0; z)$$

- E^{FV} corresponding to poles of $T^L \Leftarrow \det[1 - KG_F] = 0$, interaction-independent

$$\det[F_{l'm',lm}] = 0 \Rightarrow \begin{vmatrix} F_{\Gamma_1} & & \\ & F_{\Gamma_2} & \\ & & \ddots \end{vmatrix} = 0, \quad \det[F_{\Gamma_i}] = 0 \quad (10)$$

- Truncate at some l, reduced to irreps. of cubic group (Projection operator technique)

Lüscher's formula

- Lüscher quantization conditions: $\det\left[M^{(\Gamma,\boldsymbol{P})}_{ln,l'n'} - \delta_{ll'}\delta_{nn'}\cot\delta_l\right] = 0$

Luscher:1990ux,Rummukainen:1995vs,Feng:2004ua,Kim:2005gf,Fu:2011xz,Polejaeva:2012ut,Leskovec:2012gb,Gockeler:2012yj,...

- Example $\boldsymbol{d} = (0,0,1)$, $\Gamma = A_1^+$, w_{lm} depends on E but independent on V

$$M^{(A_1^+,\boldsymbol{d})} = \begin{bmatrix} w_{00} & -\sqrt{5}w_{20} & \cdots \\ -\sqrt{5}w_{20} & w_{00} + \frac{10}{7}w_{20} + \frac{18}{7}w_{40} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \tag{11}$$

- Truncate at $l_{max} = 0$, one-to-one relation: $\delta_0 \sim E^{FV}$
- Truncate at $l_{max} > 0$, no one-to-one relations
- One has to parameterize K-matrix: effect range expansion, unitary,..., framework-dependent
- Lüscher's formula: quantization conditions in partial wave basis
- Why not quantization conditions in plane wave basis?

Theoretical formalism

Lippmann-Schwinger equation in FV

- LSE become matrix equation $\mathbb{T} = \mathbb{V} + \mathbb{VGT}$

$$\mathbb{T}_{\boldsymbol{n}',\boldsymbol{n}} = T\left(\frac{2\pi}{L}\boldsymbol{n}', \frac{2\pi}{L}\boldsymbol{n}; E\right), \quad \mathbb{G}_{\boldsymbol{n},\boldsymbol{n}'} = \frac{1}{L^3}\frac{1}{E - \frac{q_{\boldsymbol{n}}^2}{m_N}}\delta_{\boldsymbol{n}',\boldsymbol{n}}, \quad \text{truncation at } n^2 < n_{max}^2 \tag{12}$$

- If the potential is energy-independent $\Rightarrow$ Eigenvalue problem

$$\det\left(\mathbb{G}^{-1} - \mathbb{V}\right) = 0 \rightarrow \det\left(\mathbb{H} - E\mathbb{I}\right) = 0, \quad \text{with} \quad \mathbb{H}_{\boldsymbol{m},\boldsymbol{n}} = \frac{1}{L^3}\mathbb{V}_{\boldsymbol{m},\boldsymbol{n}} + \frac{q_{\boldsymbol{n}}^2}{m_N}\delta_{\boldsymbol{m},\boldsymbol{n}}, \tag{13}$$

$$\mathbb{H} \xrightarrow{\text{reduction}} \begin{pmatrix} \mathbb{H}_{\Gamma_i} & & \\ & \mathbb{H}_{\Gamma_j} & \\ & & \ddots \end{pmatrix}_{\text{block-diagnal}}, \quad \det\left(\mathbb{H}_\Gamma - E_\Gamma\mathbb{I}\right) = 0 \tag{14}$$

Hamiltonian EFT:	PW LSE	$\rightarrow$	discretize $\lvert\boldsymbol{p}\rvert = \frac{2\pi}{L}n$	Hall:2013qba,Wu:2014vma,Liu:2015ktc, [[Li:2021mob]]
Our work:	3D LSE	$\rightarrow$	discretize $\boldsymbol{p} = \frac{2\pi}{L}\boldsymbol{n}$ $\rightarrow$	Reduce to irrep. Γ

- If the potential is E-dependent, $\det[\mathbb{M}_\Gamma(E)] = 0$, root-finding algorithm: time-consuming

Representation space

- Representation space of partial wave basis (O_h group)

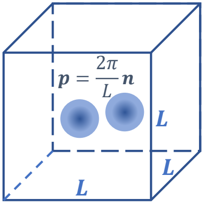
  $\Rightarrow$ representation space spanned by $|lm\rangle$ is reducible

$$\langle lm'|\hat{D}(g)|lm\rangle = \mathcal{D}^l_{m'm} \tag{15}$$

- Representation space spanned by $|\boldsymbol{p_n}\rangle$ (O_h group)

$$\{n_1, n_2, n_3\} \equiv \{|n_1, n_2, n_3\rangle \text{ + perm. } n_1, n_2, n_3 \text{ + change signs}\}$$

$$\langle \boldsymbol{n}'|\hat{D}(g)|\boldsymbol{n}\rangle = \delta_{\boldsymbol{n}', g\boldsymbol{n}}$$



- Seven patterns of representation space $\{n_1, n_2, n_3\}_{dim}$

  $\Rightarrow$ $\{0,0,0\}_1, \{0,0,a\}_6, \{0,a,a\}_{12}, \{0,a,b\}_{24}, \{a,a,a\}_8, \{a,a,b\}_{24}, \{a,b,c\}_{48}$

Representation space spanned by $|p_n\rangle$

- Reduce to irreducible representations (irreps): projection operator

e.g. textbook by M.Dresselhaus et.al

$$\hat{P}^{\Gamma_a}_{\alpha\beta} \equiv \sum_{g_i \in G} \frac{N(\Gamma_a)}{n_G} R^{\Gamma_a}_{\alpha\beta}(g_i)^* \hat{D}(g_i), \quad \hat{P}^{\Gamma_a}_{\alpha\alpha'}|\psi\rangle = a^{\Gamma_a}_{\alpha'}|\Gamma_a, \alpha\rangle. \tag{16}$$

- $R^{\Gamma_a}_{\alpha\beta}(g_i)$: unitary irrep. matrix, constructed with the character projection operators
- Examples

$\{n_1, n_2, n_3\}_D$	$\{A_1, A_2, E, T_1, T_2\}^{+\|-}$		$\{\|lm\rangle\}_{2l+1}$	$\{A_1, A_2, E, T_1, T_2\}^{+\|-}$	
$\{0,0,0\}_1$	{1,0,0,0,0}	{0,0,0,0,0}	$\|00\rangle_1$	{1,0,0,0,0}	{0,0,0,0,0}
$\{0,0,a\}_6$	{1,0,1,0,0}	{0,0,0,1,0}	$\|1m\rangle_3$	{0,0,0,0,0}	{0,0,0,1,0}
$\vdots$	$\vdots$	$\vdots$	$\|2m\rangle_5$	{0,0,1,0,1}	{0,0,0,0,0}
$\{a,b,c\}_{48}$	{1,1,2,3,3}	{1,1,2,3,3}	...	...	...

- Including spin space: e.g. $S = 1$

$$\langle \boldsymbol{p}, \boldsymbol{\eta}|\mathcal{O}_\text{p} \otimes \mathcal{O}_\text{s}|\boldsymbol{p}', \boldsymbol{\eta}'\rangle = \langle \boldsymbol{p}|\mathcal{O}_\text{p}|\boldsymbol{p}'\rangle \otimes \langle \boldsymbol{\eta}|\mathcal{O}_\text{s}|\boldsymbol{\eta}'\rangle, \quad \hat{D}(g)|\boldsymbol{\eta}\rangle = |g\boldsymbol{\eta}\rangle \tag{17}$$

General case

- Group and $\boldsymbol{d}$:$O_h : (0,0,0)$; $D_{4h} : (0,0,1)$; $D_{2h} : (1,1,0)$; $D_{3d} : (1,1,1)$...
- For $\boldsymbol{d} = (a,a,a)$: at most seven patterns

$$\{n_1, n_2, n_3\} = \{|n_1, n_2, n_3\rangle \text{ with permutations of } n_1, n_2, n_3 \text{ and changing signs}\}$$

- For $\boldsymbol{d} = (0,0,a)$ and $\boldsymbol{d} = (a,a,0)$: at most eight patterns

$$\{n_1, n_2; n_3\} = \{|n_1, n_2, n_3\rangle \text{ with permutations of } n_1 \text{and } n_2 \text{ and changing signs}\}.$$

- For elongated boxes, particles with arbitrary spin...

1. Identify the symmetric group and its elements and character table,
2. Construct the unitary irrep matrices with character projection operation,
3. Reduce the representation to a direct sum of irreps.

Root-finding and determinate residual method

- With E-dependence in potential or Lorentz transformation: $\det[\mathbb{M}_\Gamma(E)] = 0$

$$\Omega_\Gamma(E;\mu) \equiv \prod \frac{\lambda_{\Gamma,i}(E)}{\sqrt{\lambda_{\Gamma,i}(E)^2+\mu^2}}, \quad \det\left[\mathbb{M}_\Gamma(E)\right] = \prod_i \lambda_{\Gamma,i}(E)$$

- root-finding: $\Omega_\Gamma(E;\mu) = 0$

  $\Rightarrow \ -1 < \Omega_\Gamma(E;\mu) < 1;\ \ \mu$ can be chosen to optimize the root-finding procedure.

- Fitting the LQCD E^{LQCD}: spectrum method Woss:2020cmp

$$\chi^2 = \sum_{\Gamma,i} \frac{(E_{\Gamma,i} - E^{LQCD}_{\Gamma,i})^2}{\sigma(E^{LQCD}_{\Gamma,i})^2} \tag{18}$$

- Fitting the LQCD E^{LQCD}: determinant residual method Morningstar:2017spu

$$\chi^2 = \sum_{\Gamma,i} \frac{\Omega_\Gamma(E^{QCD}_{\Gamma,i})^2}{\sigma[\Omega_\Gamma(E^{QCD}_{\Gamma,i})]^2}, \tag{19}$$

Application I: spin-singlet NN scattering

Benchmark: contact interaction

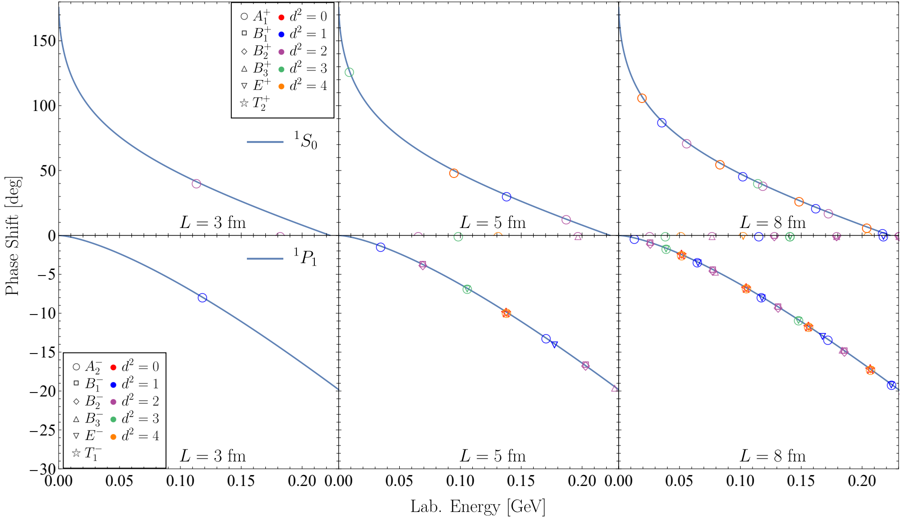
$$V^{(0)}_{\text{cont}}(\boldsymbol{p},\boldsymbol{p}') = C_S, \quad V^{(2)}_{\text{cont}}(\boldsymbol{p},\boldsymbol{p}') = C_1\boldsymbol{q}^2 + C_2\boldsymbol{k}^2 \tag{20}$$

- Non-relativistic spin singlet systems
- Short-range interaction
- ONLY contribute to S- and P-wave
- E^{FV} from the LSE of plane wave expansion in the boxes with $L = 3, 5, 8$ fm

  $\Rightarrow$ energy-independent V: Hamiltonian equation
- Moving systems: $\boldsymbol{d} = (0,0,0), (0,0,1), (1,1,0), (1,1,1), (0,0,2)$
- Extract δ^{LF} from lowest PW Lüscher formula (LF)
- Compare δ^{LF} with δ from IFV calculation

Benchmark: contact interaction

- Vanishing δ: non-interacting higher PW (Interaction:ONLY contribute to S- and P-wave)
- The single-channel Lüscher formula works accurately: short range + w/o PW mixing

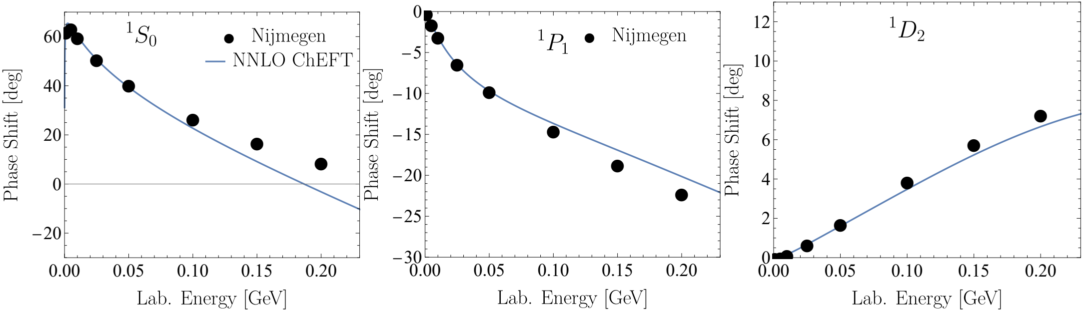


ChEFT nuclear forces to NNLO

- Interaction

Epelbaum:2003xx

$$V = V^{(0)}_{\text{cont}} + V^{(0)}_{1\pi} + V^{(2)}_{\text{cont}} + V^{(2)}_{2\pi} + V^{(2)}_{1\pi} + V^{(3)}_{2\pi} \tag{21}$$

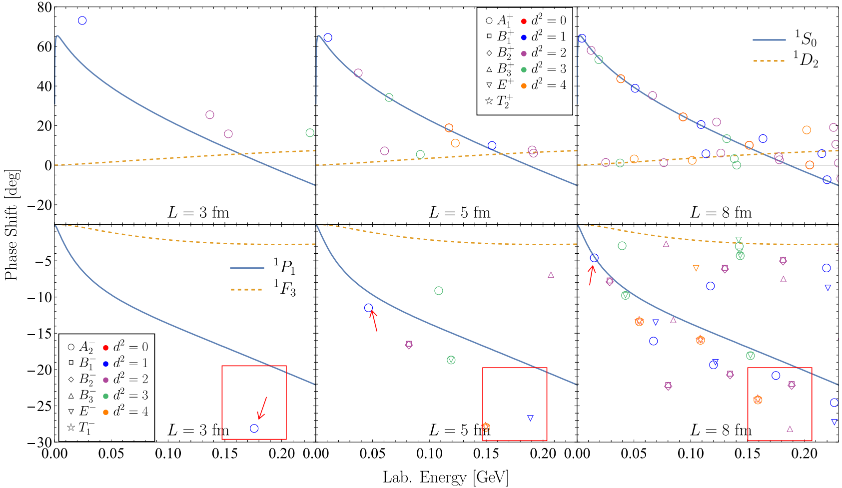


- Spin singlet: no physical partial wave mixing
- Long-range one-pion-exchange (OPE) and short range interaction

ChEFT nuclear forces: positive parity

- Large deviation for $L = 3$ fm
- Good for $L \geq 5$ fm

Sato:2007ms



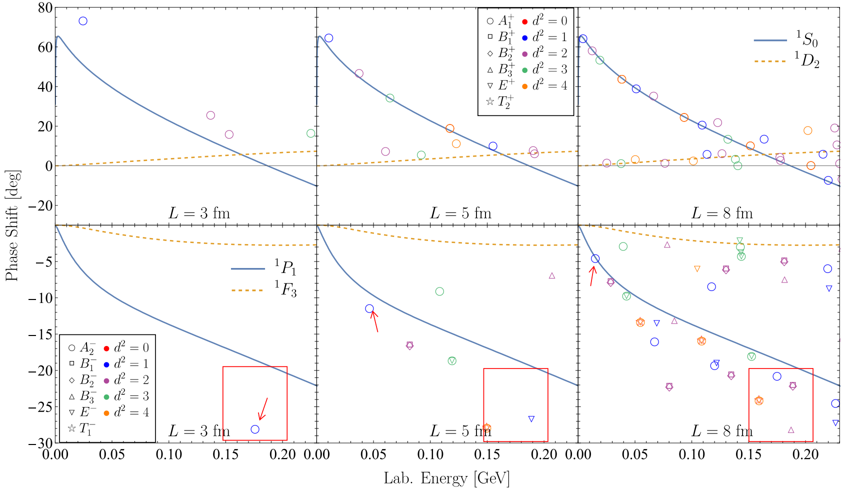
ChEFT nuclear forces: negative parity

- Large deviation even for large box: e.g. $L = 8$ fm
- Near-thresh.

  $\Rightarrow\ \delta \to$ exact ones
- Deviation$\uparrow$ with $E\uparrow$
- At higher E

  $\Rightarrow\ L\uparrow$ ~~improve~~ LF
- For single state

  $\Rightarrow\ L\uparrow$ improves LF
- PW mixing??



One-pion exchange

- For spin singlet NN

$$V_{1\pi}^{(0)}(\boldsymbol{p},\boldsymbol{p}') = \left(\frac{g_A}{2F_\pi}\right)^2 \frac{\boldsymbol{q}^2}{\boldsymbol{q}^2+M_\pi^2}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2. \tag{22}$$

- Partial wave projected OPE

$$V(\boldsymbol{p},\boldsymbol{p}') = \sum_l \frac{2l+1}{4\pi} V_l(p,p')P_l(z), \tag{23}$$

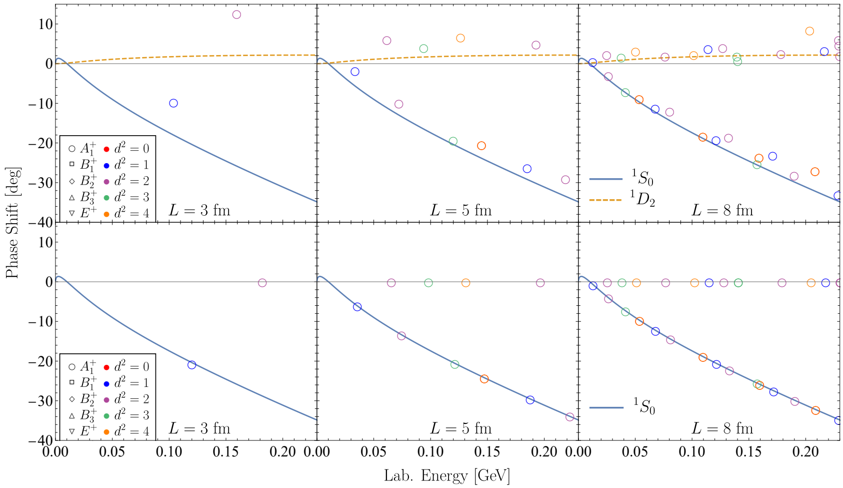
$$V_{\text{S-wave}}(\boldsymbol{p},\boldsymbol{p}') = (4\pi)^{-1}V_0(p,p')P_0(z) \tag{24}$$

$$V_{\text{P-wave}}(\boldsymbol{p},\boldsymbol{p}') = 3(4\pi)^{-1}V_1(p,p')P_1(z) \tag{25}$$

$$V_{\text{P,F-wave}}(\boldsymbol{p},\boldsymbol{p}') = 3(4\pi)^{-1}V_1(p,p')P_1(z) + 7(4\pi)^{-1}V_3(p,p')P_3(z) \tag{26}$$

One-pion exchange: positive-parity

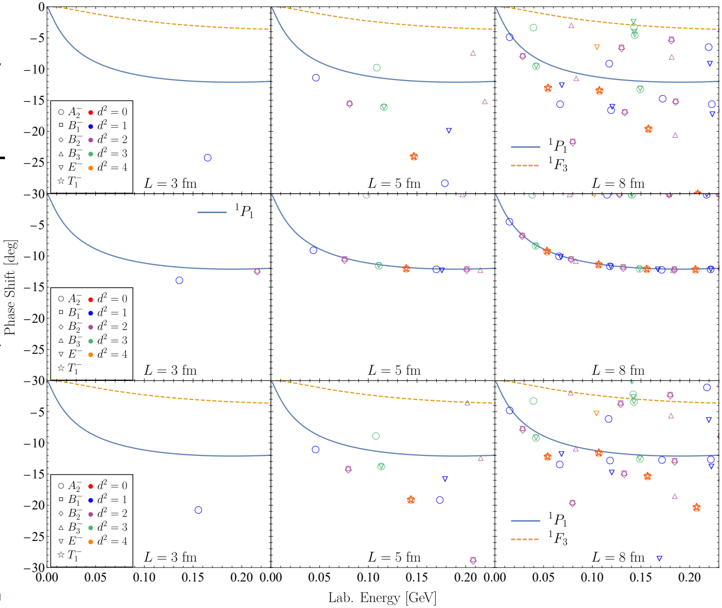
- Upper: full OPE: qualitatively similar to NNLO, large deviation for $L = 3$ fm, Good for $L \geq 5$fm
- Lower: S-wave-projected OPE: The deviation disappear



One-pion exchange: negative-parity

- The upper:full OPE
  - ⇒ Deviations are qualitatively similar to NNLO results
  - ⇒ Deviations are large regardless of L
- The middle: P-wave OPE
  - ⇒ Switch off higher PW $V_{l>1}$
  - ⇒ LF reproduces the P-wave δ accurately
- The lower: P-wave + F-wave OPE
  - ⇒ Mixing effect from F-wave
  - ⇒ Sensitivity of LF to the second lowest PW:

Lee:2021kfn



Convergence of Partial wave expansion

- Effective range expansion

$$k^{2l+1}\cot\delta_l(k) = -\frac{1}{a} + \frac{1}{2}rk^2 + \ldots \qquad (27)$$

- Near-thresh. behavior: $\delta_l(p_{\text{on}}) \sim a p_{\text{on}}^{2l+1}$
- ONLY below the lowest t-channel singularity: $E_{\text{lab}} \sim 2M_\pi^2/m_N \sim 10$ MeV Baru:2015ira
- Above this energy: convergence of the partial wave expansion becomes slow

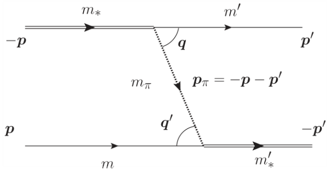
  $\Rightarrow$ NN differential cross section at $E_{\text{lab}} = 300$ MeV at the 1% accuracy level: $j_{\text{max}} = 16$

  Fachruddin:2001az

- Long-range (OPE) interaction: physical m_π

  $\Rightarrow$ one-to-one LF fails due to PW mixing

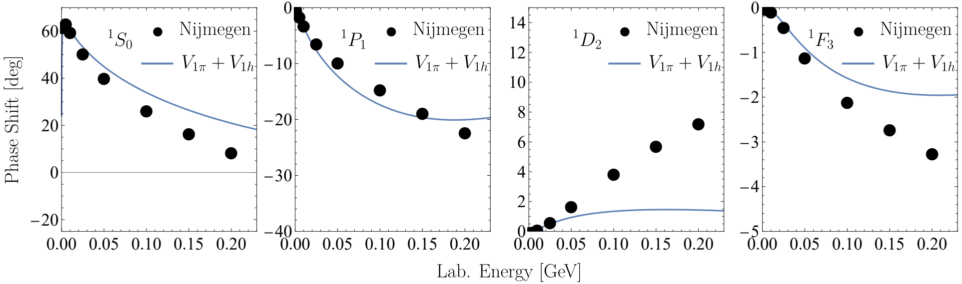
  $\Rightarrow$ $\bar{D}^* D$ system OPE: on-shell pion



EFT-inspired fitting: toy model

- Long-range (OPE) interaction ⇒one-to-one LF fails due to PW mixing
- Known OPE⇒ EFT-inspired approach to fitting E^{FV}
- Mimic LQCD data: Toy model to generate FV energies, m_h=0.5 GeV

$$V_{\text{toy}} = V_{1\pi} + V_{1h} = -\left(\frac{g_A}{2F_\pi}\right)^2 \frac{M_\pi^2}{\boldsymbol{q}^2 + M_\pi^2}\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 + (c_{h1} + c_{h2}\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)\frac{1}{\boldsymbol{q}^2 + m_h^2} \tag{28}$$



EFT-inspired fitting

- Fitting interaction

$$V_{\text{EFT}} = V^{(0)}_{\text{OPE}} + V^{(0)}_{\text{cont}} + V^{(2)}_{\text{cont}} + V^{(4)}_{\text{cont}} + \dots \tag{29}$$

- Long-range part

$$V^{(0)}_{\text{OPE}} = -\left(\frac{g_A}{2F_\pi}\right)^2 \frac{M_\pi^2}{\boldsymbol{q}^2 + M_\pi^2} \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \tag{30}$$

- Short range part

$$V^{(0)}_{\text{cont}} = \frac{1}{4\pi}\tilde{C}_{^1S_0}, \quad V^{(2)}_{\text{cont}} = \frac{1}{4\pi}C_{^1S_0}(p^2 + p'^2) \tag{31}$$

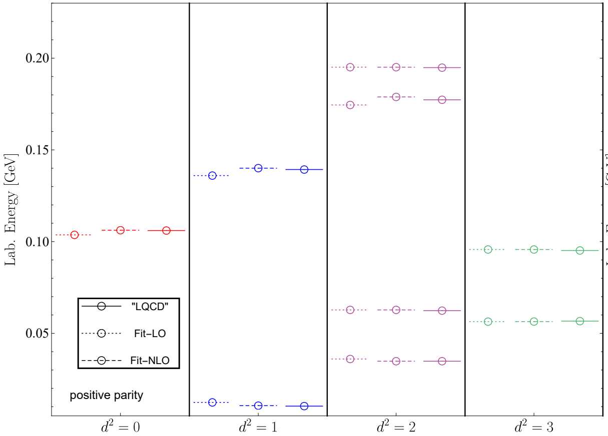
$$V^{(2)}_{\text{cont}}(p, p', z) = \frac{3}{4\pi}C_{^1P_1}pp'z, \quad V^{(4)}_{\text{cont}}(p, p', z) = \frac{3}{4\pi}D_{^1P_1}pp'(p^2 + p'^2)z \tag{32}$$

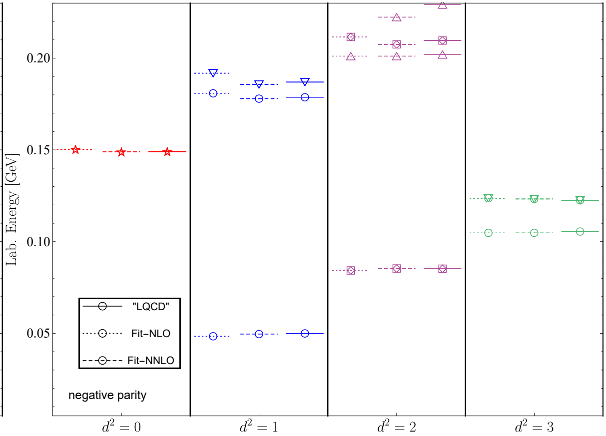
- Fitting: Determinant residual method

Morningstar:2017spu

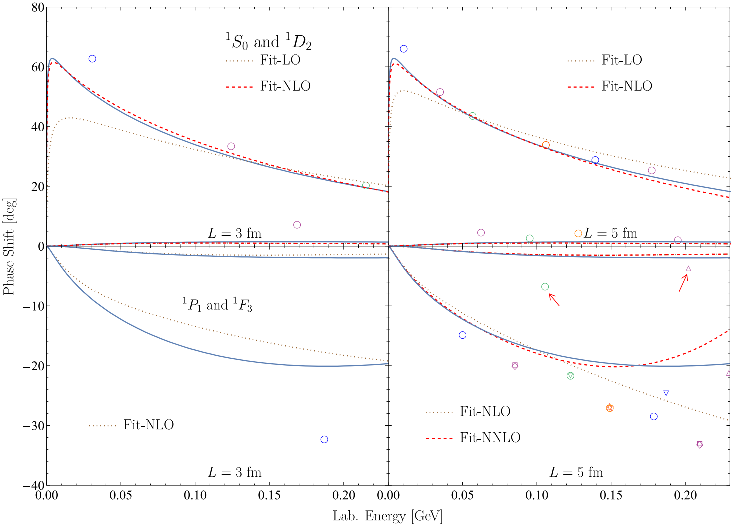
EFT-inspired fitting: results

- The ground state for each irrep is input
- Good agreement and improved with orders





EFT-inspired fitting VS Lüscher formula



- Improved with orders
  - $\Rightarrow$ 1-para.: rough
  - $\Rightarrow$ 2-para: improved
- Uncover "underlying" theory
- Good fit for small box
  - $\Rightarrow$ e.g. S-wave, $L = 3$ fm
- Higher PW dominant energy will NOT fail the fit

Application II: ρ-channel $\pi\pi$ scattering

- Reduced Bethe-Salpeter equation

Woloshyn:1973mce

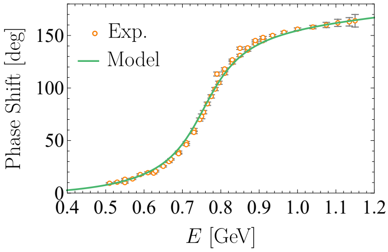
$$T(\boldsymbol{p},\boldsymbol{p}';E) = V(\boldsymbol{p},\boldsymbol{p}';E) + i\int_{q<q_{\text{max}}} \frac{d^3\boldsymbol{q}}{(2\pi)^3} V(\boldsymbol{p},\boldsymbol{q};E)G(q;E)T(\boldsymbol{q},\boldsymbol{p}';E). \tag{33}$$

$$G(q,E) = \frac{1}{2\omega_1(q)\omega_2(q)} \frac{\omega_1(q)+\omega_2(q)}{E^2-[\omega_1(q)+\omega_2(q)]^2+i\epsilon} \tag{34}$$

- Phenomenological model:

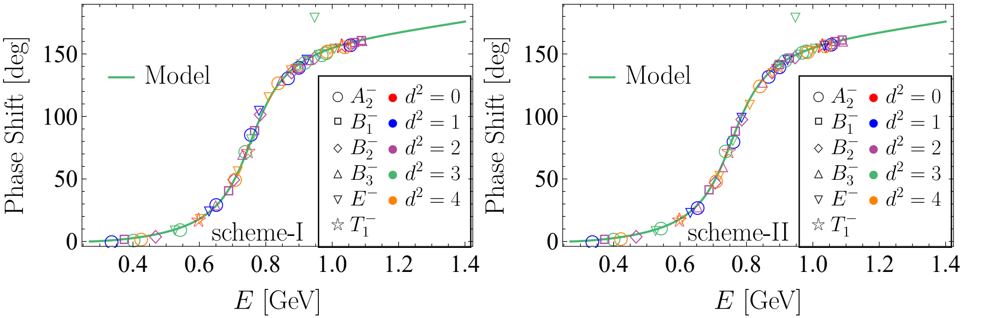
Chen:2012rp

$$V(\boldsymbol{p},\boldsymbol{p}';E) = -\frac{2\boldsymbol{p}\cdot\boldsymbol{p}'}{f^2}\left(1+\frac{2G_V^2}{f^2}\frac{E^2}{M_0^2-E^2}\right) \tag{35}$$

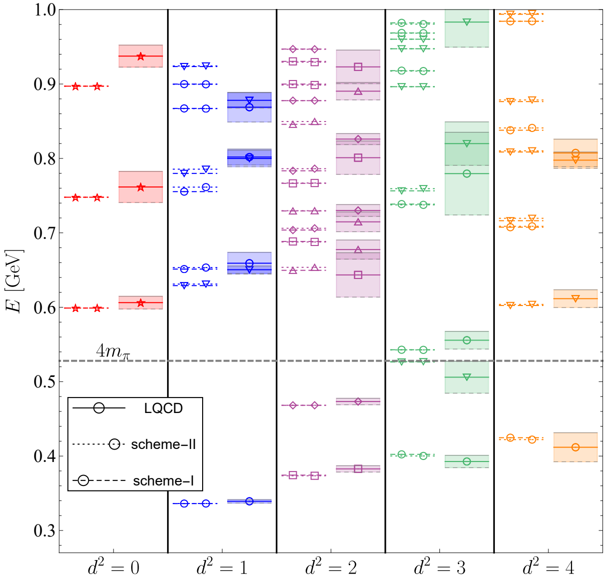


- ONLY P-wave
- 3 paras. (f, G_V and M_0) depict the δ in IFV

- Plane wave expansion of BSE in FV ($L = 4.3872$ fm)
- Energy-dependent potential$\Rightarrow$ root-finding algorithm
- Lüscher formula: extract the phase shift from energy levels
- Lüscher formula works well (short range interaction without PW mixing)
- The difference of two schemes is very tiny



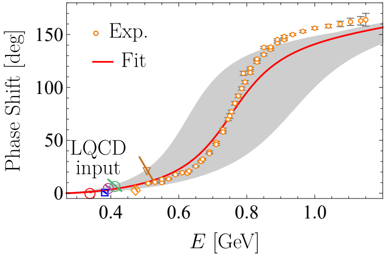
Compare with and fit the lattice QCD results



- Lattice QCD results from ETMC Fischer:2020fvl

  $\Rightarrow L = 4.3872,\ m_\pi = 132$ MeV

- Compare our E^{FV} with LQCD results
- Fit the E^{LQCD} below $4m_\pi$
- Agree with Exp. results well
- Large uncertainty due to simple model



Summary

Summary

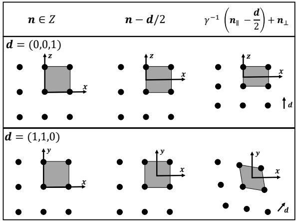
- A proof-of-principle of an alternative method of Lüscher's formula
- LSE or BSE in plane wave expansion+projection operator technique reduction to irreps
  - ⇒ Including partial wave mixing effect naturally, avoid complications of PW expansion
  - ⇒ Rest and moving two-particle systems, spinless, equal mass
- Non-relativistic example: spin-singlet NN
  - ⇒ S-wave dominant states: LF works well for $L \gtrsim 5$ fm
  - ⇒ P-wave dominant states: OPE→large PW mixing effect regardless the box size
  - ⇒ EFT-based approach in the plane wave basis: $V_{1\pi} + V_{1h} \underset{\text{data}}{\overset{\text{fit}}{\rightleftharpoons}} V_{1\pi} + V^{(0)}_{\text{cont}} + V^{(2)}_{\text{cont}} \cdots$
  - ⇒ Advantages: 1) insensitive to PW mixing artifact; 2) small box (long-range interaction)
- Relativistic example: ρ-channel $\pi\pi$: compare and fit with LQCD results from ETM
- Straightforward to nonzero spin, particles with different masses, elongated boxes

Thanks for your attention!

Backup

Projection operator technique

- Identify the symmetric group and its elements and character table



- Construct the unitary irrep matrices with character projection operation

$$\hat{P}^{\Gamma_a} \equiv \sum_{\alpha} \hat{P}^{\Gamma_a}_{\alpha\alpha} = \sum_{g_i \in G} \frac{N(\Gamma_a)}{n_G} \chi^{\Gamma_a}(g_i) \hat{D}(g_i), \quad \hat{P}^{\Gamma_a}|\psi\rangle = \sum_{\alpha} a^{\Gamma_a}_{\alpha} |\Gamma_a, \alpha\rangle$$

- Reduce the representation to the direct sum of irreps.

$$\hat{P}^{\Gamma_a}_{\alpha\beta} \equiv \sum_{g_i \in G} \frac{N(\Gamma_a)}{n_G} R^{\Gamma_a}_{\alpha\beta}(g_i)^* \hat{D}(g_i), \quad \hat{P}^{\Gamma_a}_{\alpha\alpha'}|\psi\rangle = a^{\Gamma_a}_{\alpha'} |\Gamma_a, \alpha\rangle.$$

$$V(\boldsymbol{p},\boldsymbol{p}') = \sum_l \frac{2l+1}{4\pi} V_l(p,p') P_l(z)$$

$$T(\boldsymbol{p},\boldsymbol{p}') = V(\boldsymbol{p},\boldsymbol{p}') + \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} V(\boldsymbol{p},\boldsymbol{q}) G(q) T(\boldsymbol{q},\boldsymbol{p}')$$

$$T^l(p,p') = V^l(p,p') + \int \frac{q^2 dq}{(2\pi)^3} V^l(p,q) G(q) T^l(q,p')$$

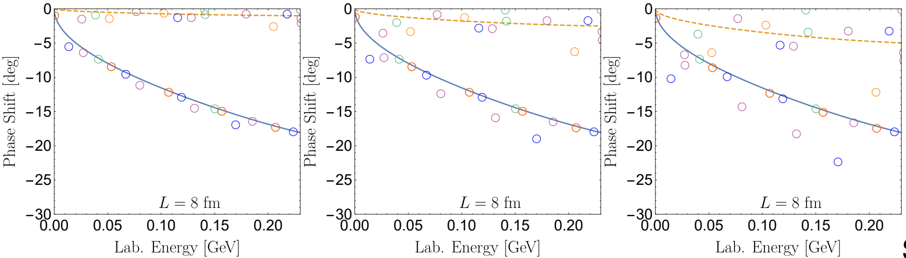
- For S-wave: $V_0(p,p') \neq 0$ and $V_l(p,p') = 0$ for $l > 0$; two approaches are the same
- For the higher PW, e.g.: $V(\boldsymbol{p},\boldsymbol{p}') = \frac{3}{4\pi} V_1(p,p') P_1(z)$

  $\Rightarrow$ The Adelaide group in fact assume a

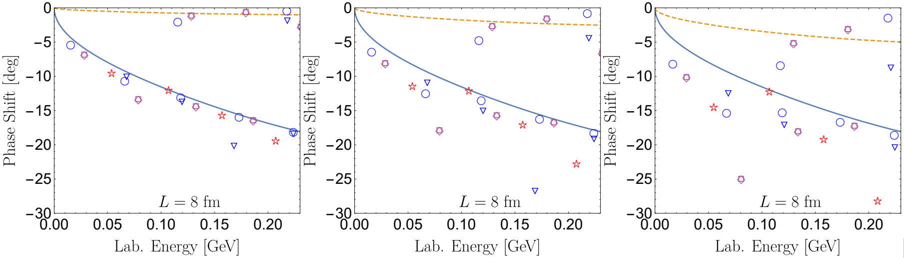
$$V(\boldsymbol{p},\boldsymbol{p}') = \frac{1}{4\pi} \tilde{V}_0(p,p') = \frac{1}{4\pi} V_1(p,p')$$

PW mixing

$$V(p,p',z) = \sum_l \frac{l+1}{4\pi} V_l P_l(z), \quad V_{l=0} = V_{l=1} = C_1, \quad V_{l=2} = V_{l=3} = C_2, \quad C_2 = 10c_2, 15c_2, 50c_2$$

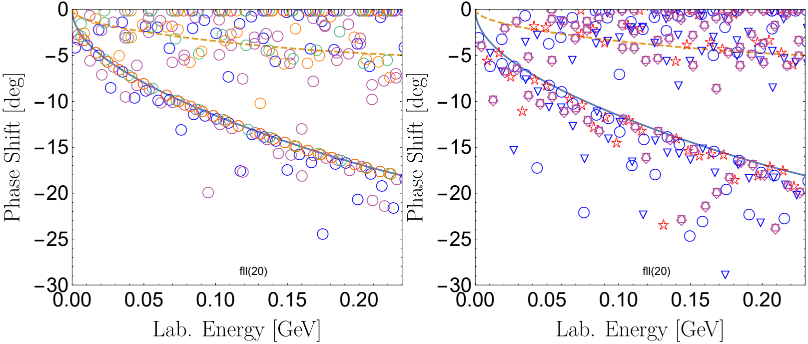


S-D mixing



P-F mixing

PW mixing: $L = 20$ fm



Computational cost- an estimation

- For NN: $n^2 \leq 100$, dim $\approx \frac{4}{3}\pi n^3 \approx 4000$, $\mathrm{dim}_\Gamma \approx 4000/10$
- For $\pi\pi$: $n^2 \leq 50$, dim $\approx \frac{4}{3}\pi n^3 \approx 1500, \mathrm{dim}_\Gamma \approx 1500/10$
- Accelerating the calculation: eigenvector continuation??

Frame:2017fah

Luscher formula

$$\langle \boldsymbol{p}|T^L(z)|\boldsymbol{q}\rangle = \langle \boldsymbol{p}| - V(z)|\boldsymbol{q}\rangle + \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\langle \boldsymbol{p}| - V|\boldsymbol{k}\rangle G_0^L(\boldsymbol{k};z)\langle \boldsymbol{k}|T^L|\boldsymbol{q}\rangle$$

$$G_0^L(\boldsymbol{k},z) = (\frac{2\pi}{L})^3 \sum_{\boldsymbol{p}} \frac{2\mu\delta^3(\boldsymbol{p}-\boldsymbol{k})}{\boldsymbol{p}^2 - q_0^2} = G_K(\boldsymbol{k},z) + G_F(\boldsymbol{k},z)$$

$$G_K(\boldsymbol{k},z) = 2\mu\mathcal{P}\frac{1}{\boldsymbol{k}^2 - q_0^2}, \quad T^L = K + KG_FT$$

$$z = m_1 + m_2 + \frac{q_0}{2\mu}$$

$$T^L = K + KG_FT^L$$

$$\langle \boldsymbol{p} | T^L | \boldsymbol{q} \rangle = 4\pi \sum_{l'm'lm} Y_{l'm'}(\hat{\boldsymbol{p}}) T^L_{l'm',lm}(p,q,z) Y^*_{lm}(\hat{\boldsymbol{q}})$$

$$\langle \boldsymbol{p} | K | \boldsymbol{q} \rangle = 4\pi \sum_{lm} Y_{lm}(\hat{\boldsymbol{p}}) K_l(p,q,z) Y^*_{lm}(\hat{\boldsymbol{q}})$$