High-precision determination of the electric and magnetic radius of the proton

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Outline

- Proton charge radius
- Dispersion theoretical determination of r_p^E
  - Parametrization of nucleon FFs
  - Application to data
  - Results and uncertainty
- Status of proton radius puzzle
- Summary

Proton charge radius

- Definition

$$G_{\mathrm{E}}(Q^2) = 1 - \frac{r_p^2}{3!}Q^2 + \frac{\langle r^4 \rangle_{\mathrm{E}}}{5!}Q^4 - \frac{\langle r^6 \rangle_{\mathrm{E}}}{7!}Q^6 + \ldots$$

$$r_p^2 = -6 \left. \frac{dG_E(Q^2)}{d\,Q^2} \right|_{Q^2=0}$$

- Measurement C. Peset, *et al.* arXiv2106.00695
  - Leptonic hydrogen Lamb shift

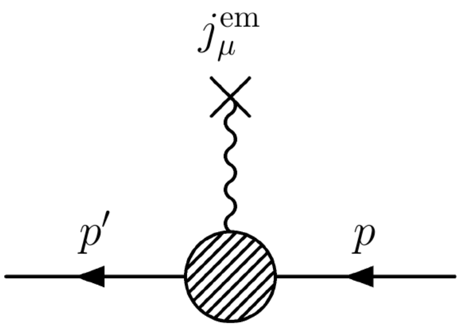
$$(\Delta E_L)_{\text{measured}} = E_1 + E_2 C(r_p^2) + \mathcal{O}(m_r \alpha^6), \quad C(r_p^2) = c_1 + c_2 r_p^2 + \mathcal{O}(\alpha^2)$$

  - Lepton-proton Scattering

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{measured}} = \frac{d\sigma_{\text{Mott}}}{d\Omega}\frac{1}{1+\tau_p}\left(G_E^2 + \frac{\tau_p}{\varepsilon}G_M^2\right)(1+\delta_{\text{TPE}}) + \mathcal{O}(\alpha^2)$$

Nucleon Form Factors

- Definition



$$\langle p'|j_\mu^{\text{em}}|p\rangle = \bar{u}(p')\left[F_1(t)\gamma_\mu + i\frac{F_2(t)}{2m}\sigma_{\mu\nu}q^\nu\right]u(p)\,,$$

$$t \equiv q^2 = -Q^2 = (p'-p)^2,\ \ t > 0 \text{ for time-like, } t < 0 \text{ for space-like}$$

- Normalization $F_1^p(0) = 1\,, F_1^n(0) = 0\,,\ F_2^p(0) = \kappa_p\,, F_2^n(0) = \kappa_n\,.$

- Isoscalar & isovector NFFs

$$F_i^s = \frac{1}{2}(F_i^p + F_i^n)\,, F_i^v = \frac{1}{2}(F_i^p - F_i^n)\,, i = 1, 2$$

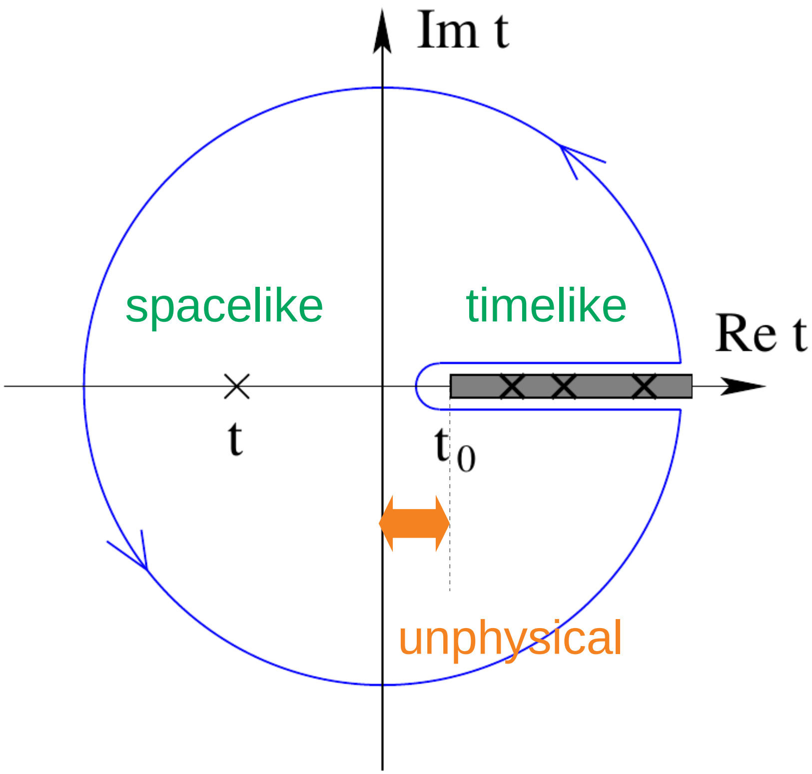
- Sachs NFFs

$$G_E(t) = F_1(t) - \tau F_2(t)\,, G_M(t) = F_1(t) + F_2(t)$$

Why Dispersion Theory?

- Difficulties on NFFs
  - Unknown expression $\longrightarrow$ parametrization dependent
  - Data at $Q^2 = 0$ is unachievable $\longrightarrow$ extrapolation needed
- Dispersion theoretical NFFs $F(t) = \frac{1}{\pi} \int_{t_0}^{\infty} \frac{\mathrm{Im}\, F(t')}{t' - t - i\epsilon}\, dt'$
  - Unitarity and analyticity guaranteed,
  - Works well in whole energy region, $(\sim 10^{-4}\text{-}10\mathrm{s\, GeV}^2)$ experimentally
  - Theoretical constrains of asymptotic behavior of NFFs can be added easily,
  - Connects to data from different process. $(\pi N\text{-scattering}, \cdots)$

Dispersion Relations of NFFs



$$F(t) = \frac{1}{\pi} \int_{t_0}^{\infty} \frac{\mathrm{Im}\, F(t')}{t' - t - i\epsilon} \, dt'$$

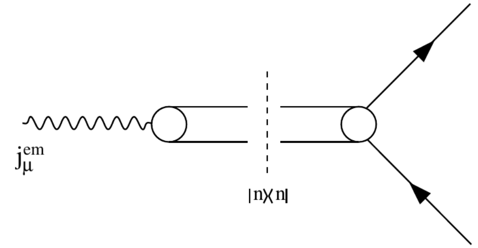
Ingredients: multiple cuts
(start from t_0)
&& vector meson poles

The spectral function $\mathrm{Im} F(t)$ are central quantities.

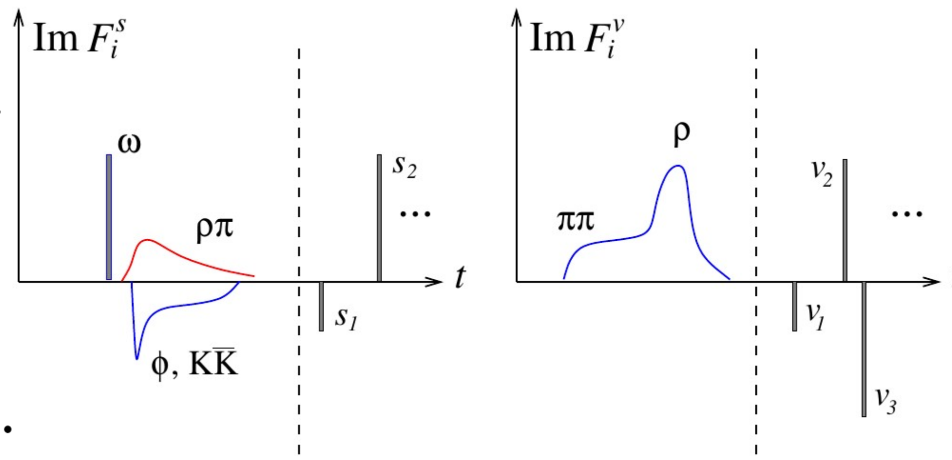
Spectral functions

- Spectral Decomposition (lower energy part)
  - Crossing symmetry $\langle N(p')|j_\mu^{\rm em}|N(p)\rangle \longleftrightarrow \langle N(p)\bar{N}(\bar{p})|j_\mu^{\rm em}|0\rangle$
  - Spectral decomposition G. F. Chew, *et al.* PhysRev110, 265(1958)

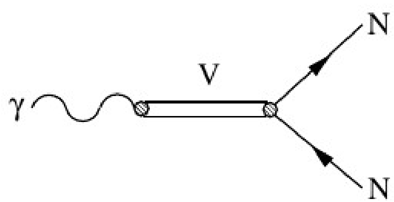
$$\mathrm{Im}\langle N(p)\bar{N}(\bar{p})|j_\mu^{\rm em}|0\rangle \sim \sum_n \langle N(p)\bar{N}(\bar{p})|n\rangle\langle n|j_\mu^{\rm em}|0\rangle$$



  - Intermediate mass states $|n\rangle$
    - Isoscalar: $3\pi, 5\pi, \ldots, K\bar{K}, \pi\rho, \ldots$
    - Isovector: $2\pi, 4\pi, \ldots$



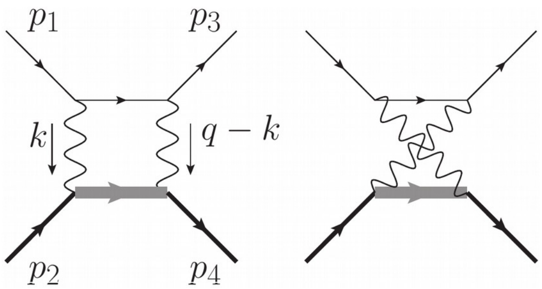
- Vector meson dominance



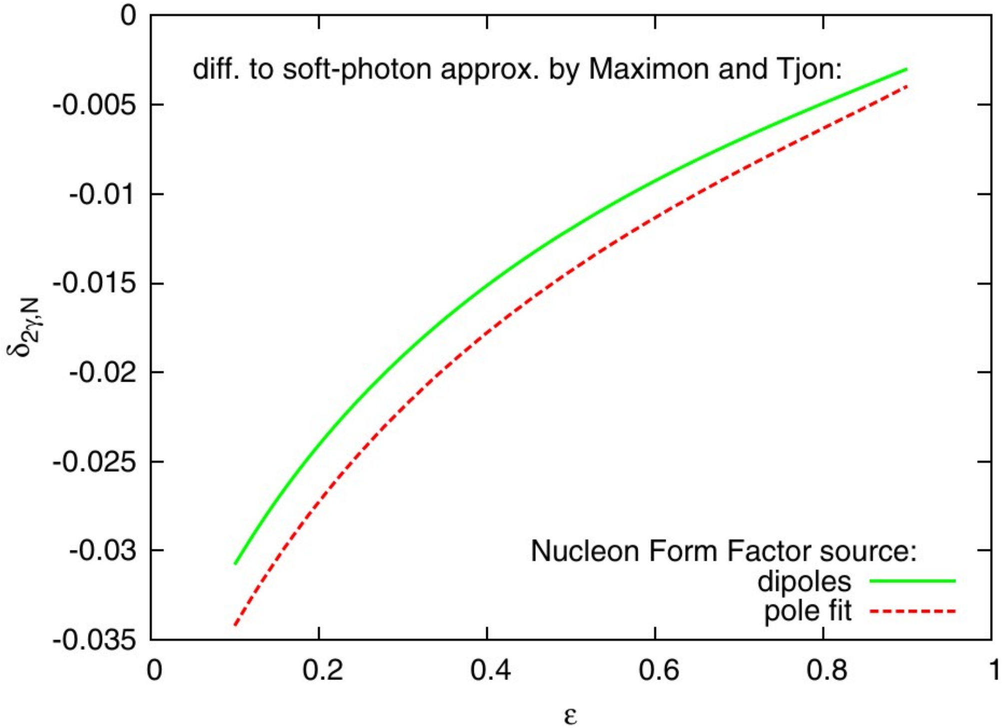
Higher mass state $s_1, s_2, \ldots$
$v_1, v_2, \ldots$

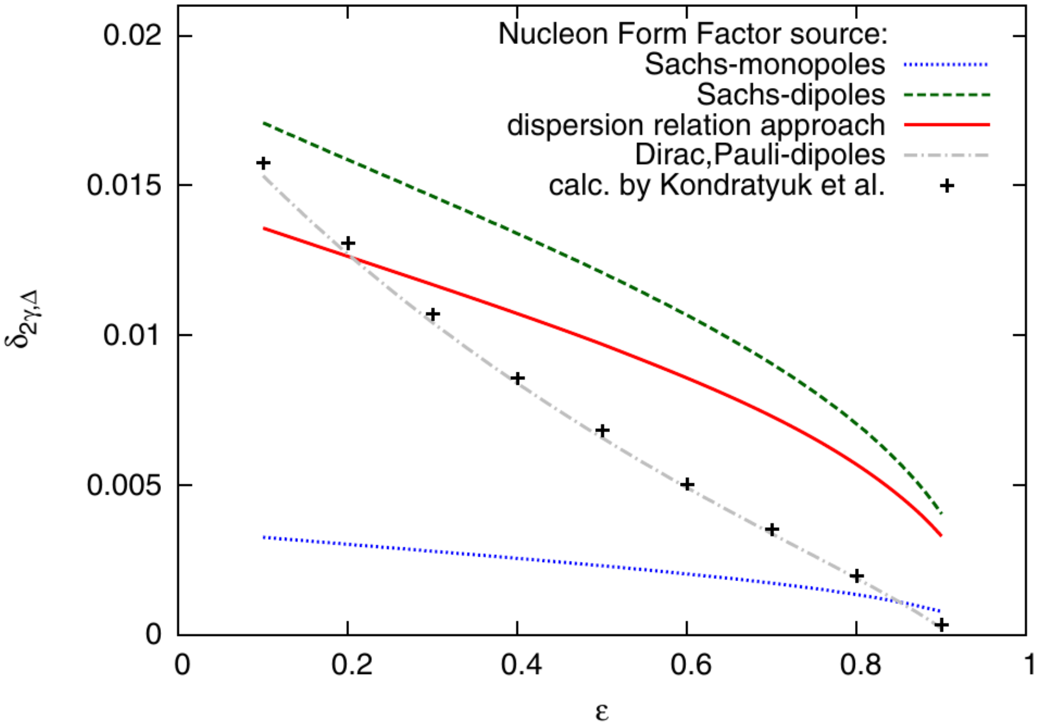
Two-photon effects

- Diagram



- Numerical result for $\delta_{2\gamma,N}$, $\delta_{2\gamma,\Delta}$





Two-photon effects

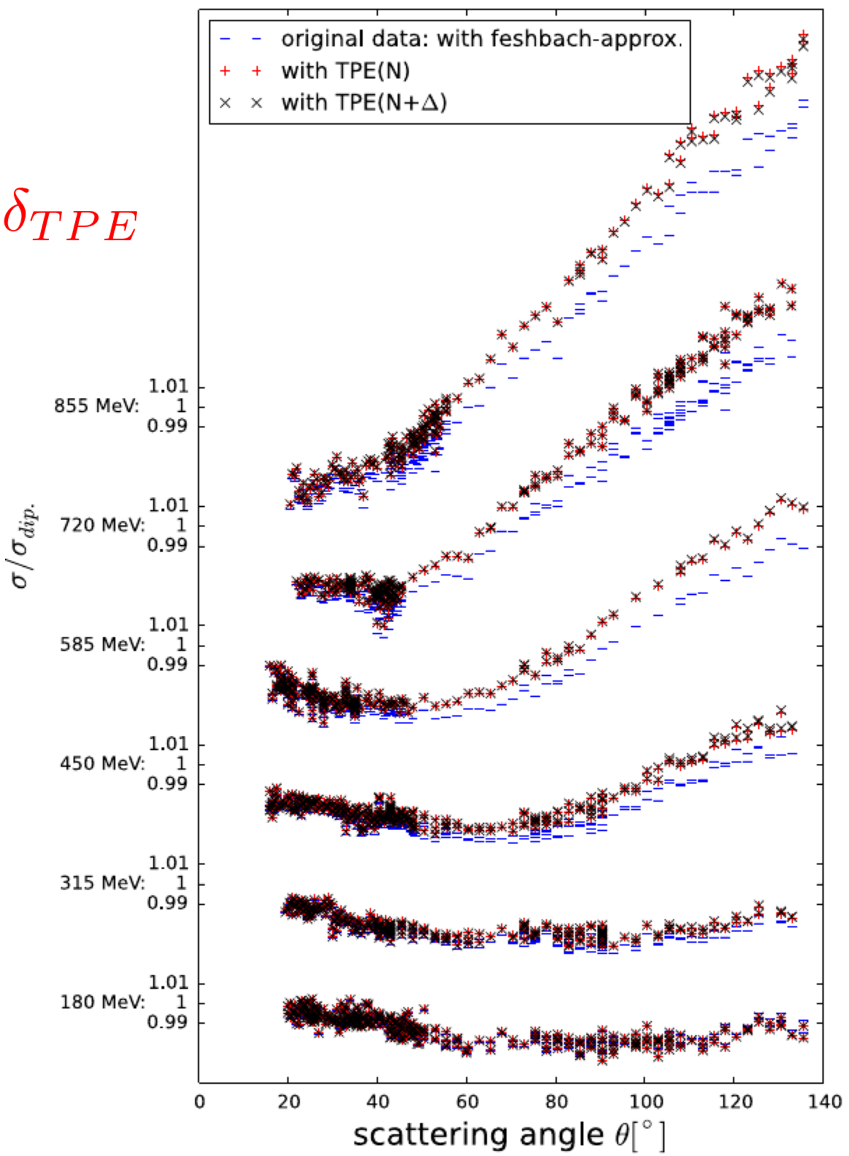
- Correction to XS

$\text{smaller}\,\theta \longrightarrow \text{lower}\,Q^2 \longrightarrow \text{smaller}\,\delta_{TPE}$

A simple form for $\delta_{2\gamma}$ that only works in the limit $Q^2 \to 0$ contained in MAMI data.

$$\delta_F = Z\alpha\pi\frac{\sin(\theta/2) - \sin^2(\theta/2)}{\cos^2(\theta/2)}$$

W. A. McKinley, *et al.* PhysRev74, 1759(1948)



Theoretical constrains

- Normalization (4)
- Neutron charge radius squared (1)

A. A. Filin, *et al.* PhysRevLett124, 082501(2020)

$$\langle r_n^2 \rangle = -0.105^{+0.005}_{-0.006}\ \mathrm{fm}^2$$

- pQCD asymptotic behavior of NFFs (6)

$$\int_{t_0}^{\infty} \mathrm{Im}\, F_i(t)\, t^n dt = 0\,, \quad i = 1, 2$$

with $n = 0$ for F_1, $n = 0, 1$ for F_2

Data bases

- Differential cross section
  - MAMI (0.00384-0.977 GeV^2, 1422)
  - PRad (0.000215-0.058, 71)
- World data on Neutron form factor
  - G_E^n (0.14-1.47, 25)
  - G_M^n (0.071-10.0, 23)
- JLab data on Proton FFs ratio

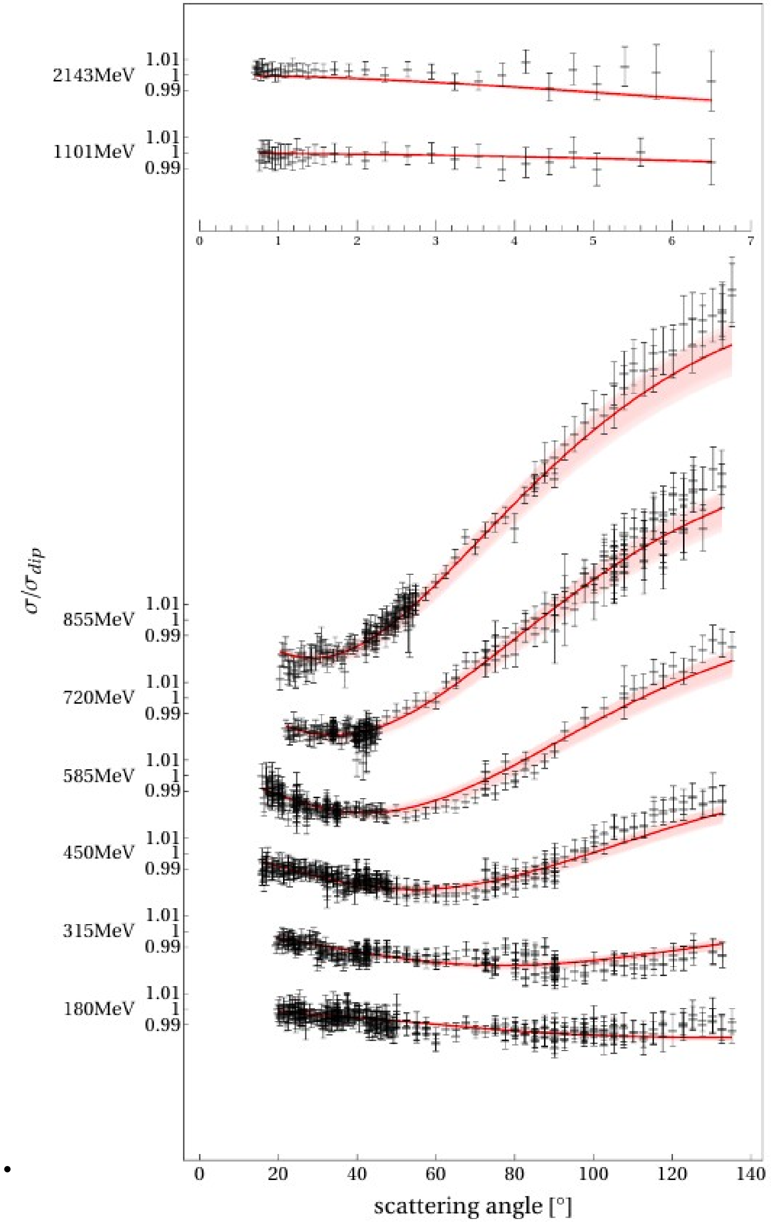
  $\mu_p G_E^p / G_M^p$ (1.18-8.49, 16)

- Number of free parameter

$$1557 \Longrightarrow 4 + 3(N_s + N_v) - 11 + 31 + 2$$

Results I: Differential cross section

- Best configuration '6s+4v'
  - 50 parameters
  - $\omega, \phi, s_1, s_2, s_3, s_4 + K\bar{K} + \rho\pi$
  - $v_1, v_2, v_3, v_4 + \pi\pi$
  - $\chi^2/\text{dof} = 1.927$

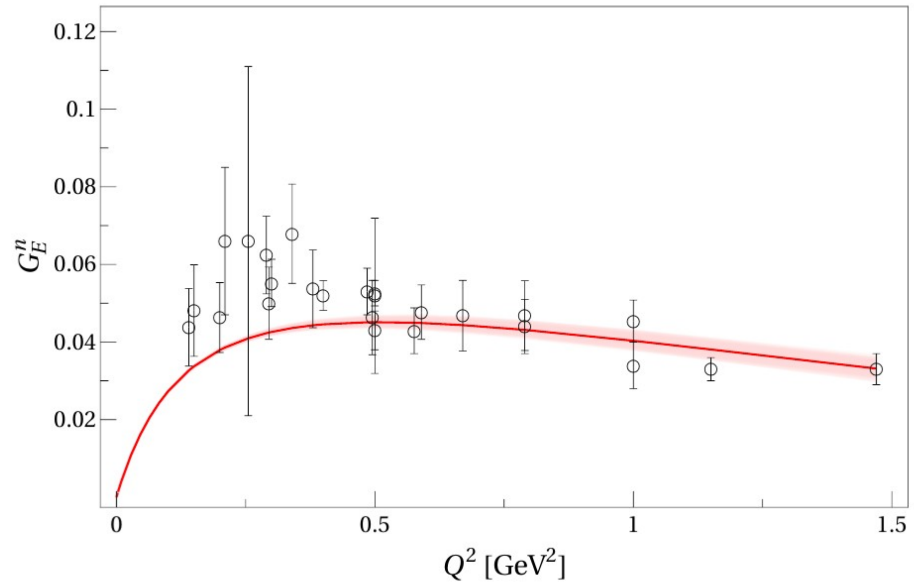


Line best fit

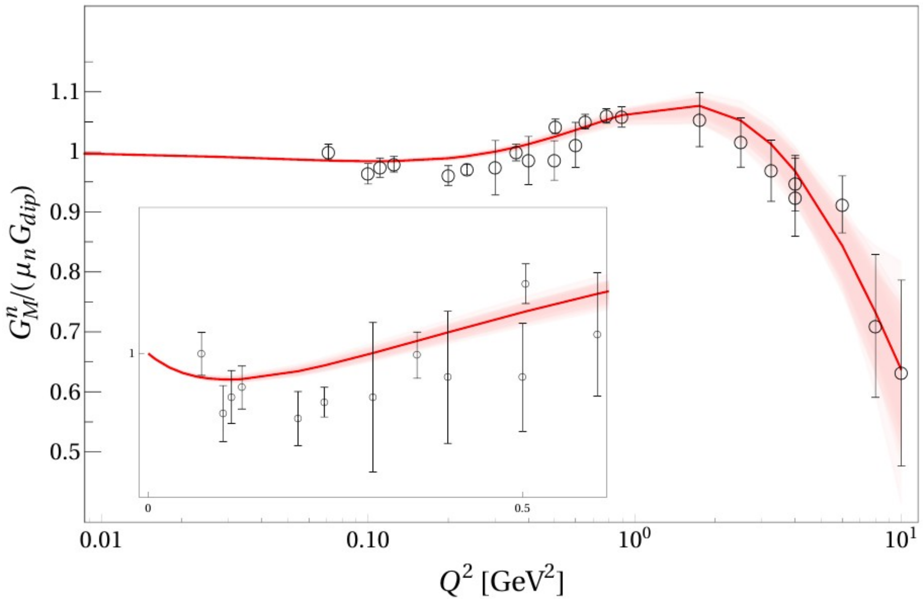
Shadow error band from bootstrap sampling.

Results II: NFFs

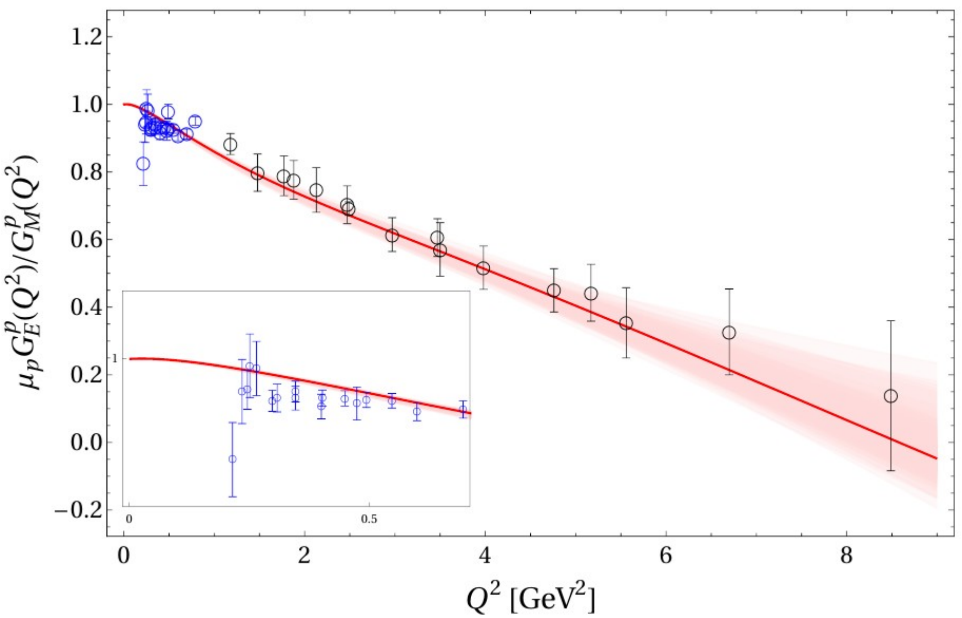
G_E^n



$G_M^n/(\mu_n G_{\mathrm{dip}})$



$\mu_p G_E^p / G_M^p$



Line best fit

Shadow error band from bootstrap sampling.

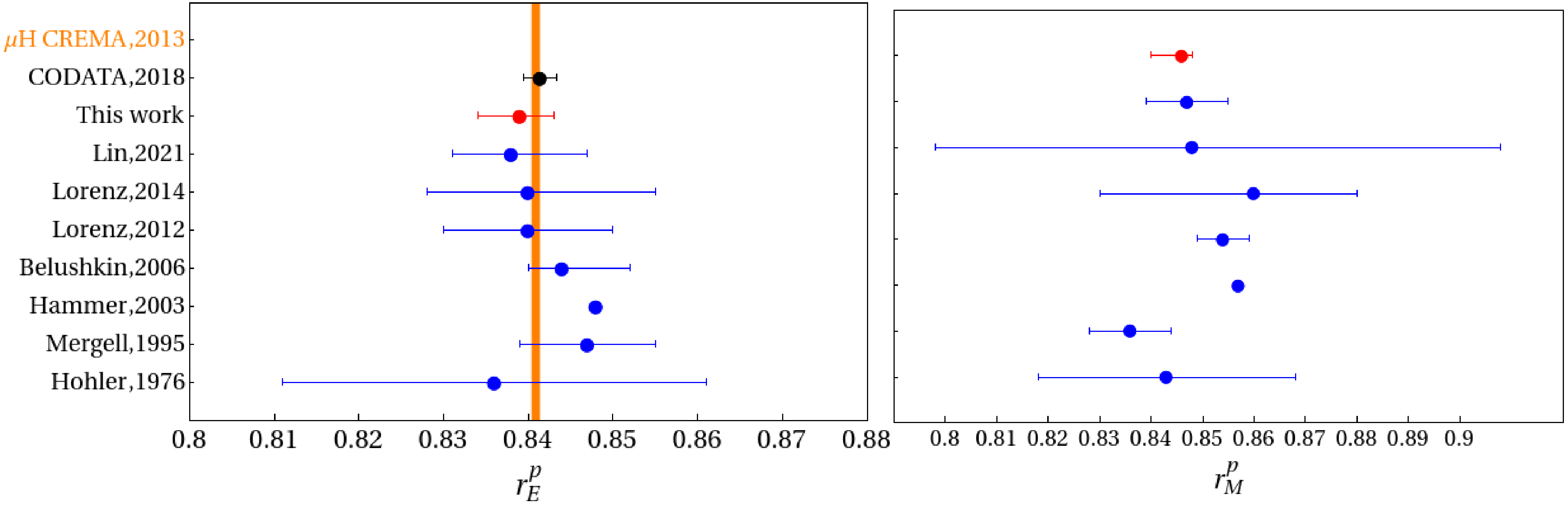
Focusing on proton charge radius

- Our determination (statistical error form bootstrap)

  Systematical error from variation of spectral function

$$r_E^p = 0.839 \pm 0.002^{+0.002}_{-0.003}\,\text{fm},\ r_M^p = 0.846 \pm 0.001^{+0.001}_{-0.005}\ \text{fm}$$

- Comparing to existing DR determination

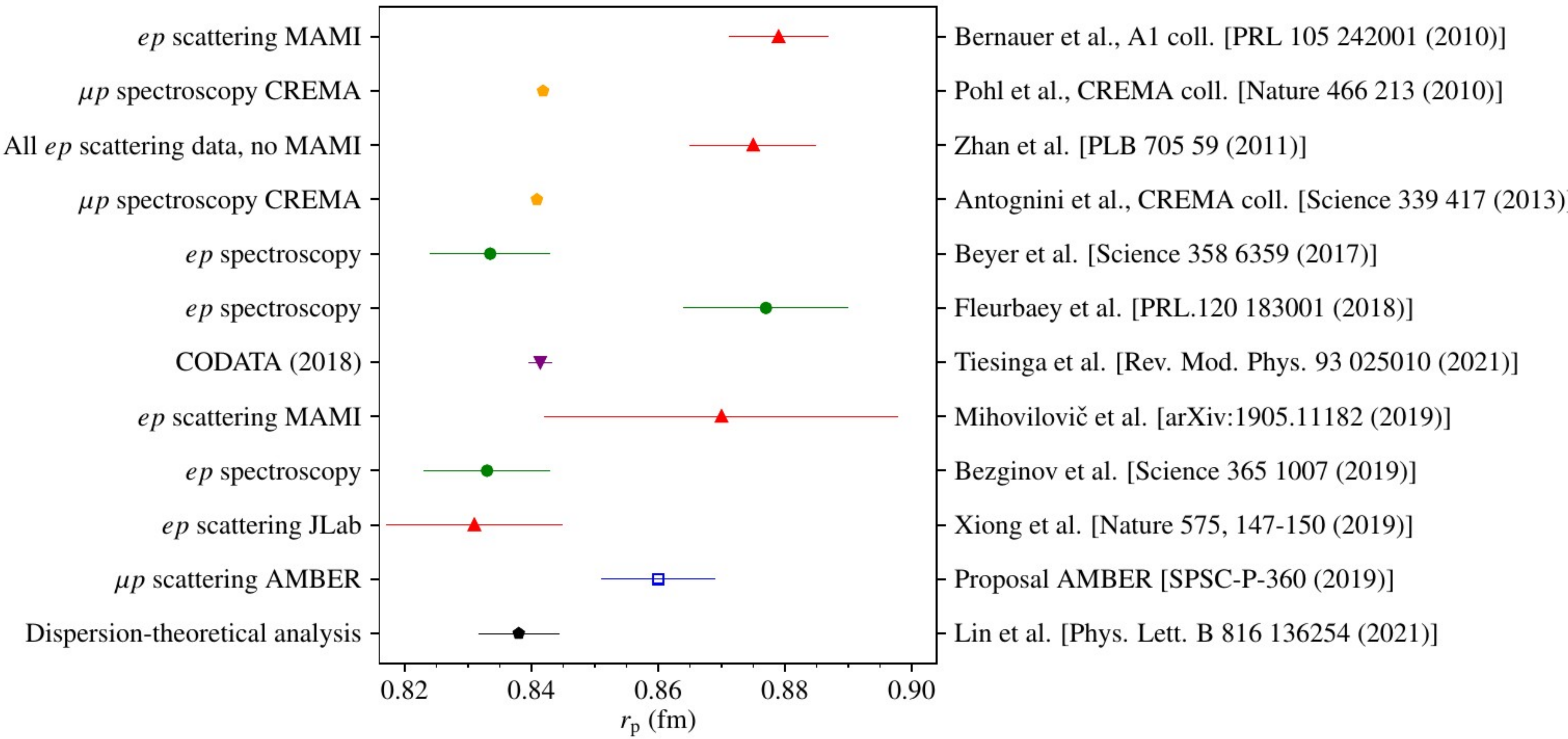


Comparing to newest measurement

Fundamental Physical Constants

proton rms charge radius r_p	
Numerical value	8.414 x 10^-16 m
Standard uncertainty	0.019 x 10^-16 m
Relative standard uncertainty	2.2 x 10^-3
Concise form	8.414(19) x 10^-16 m

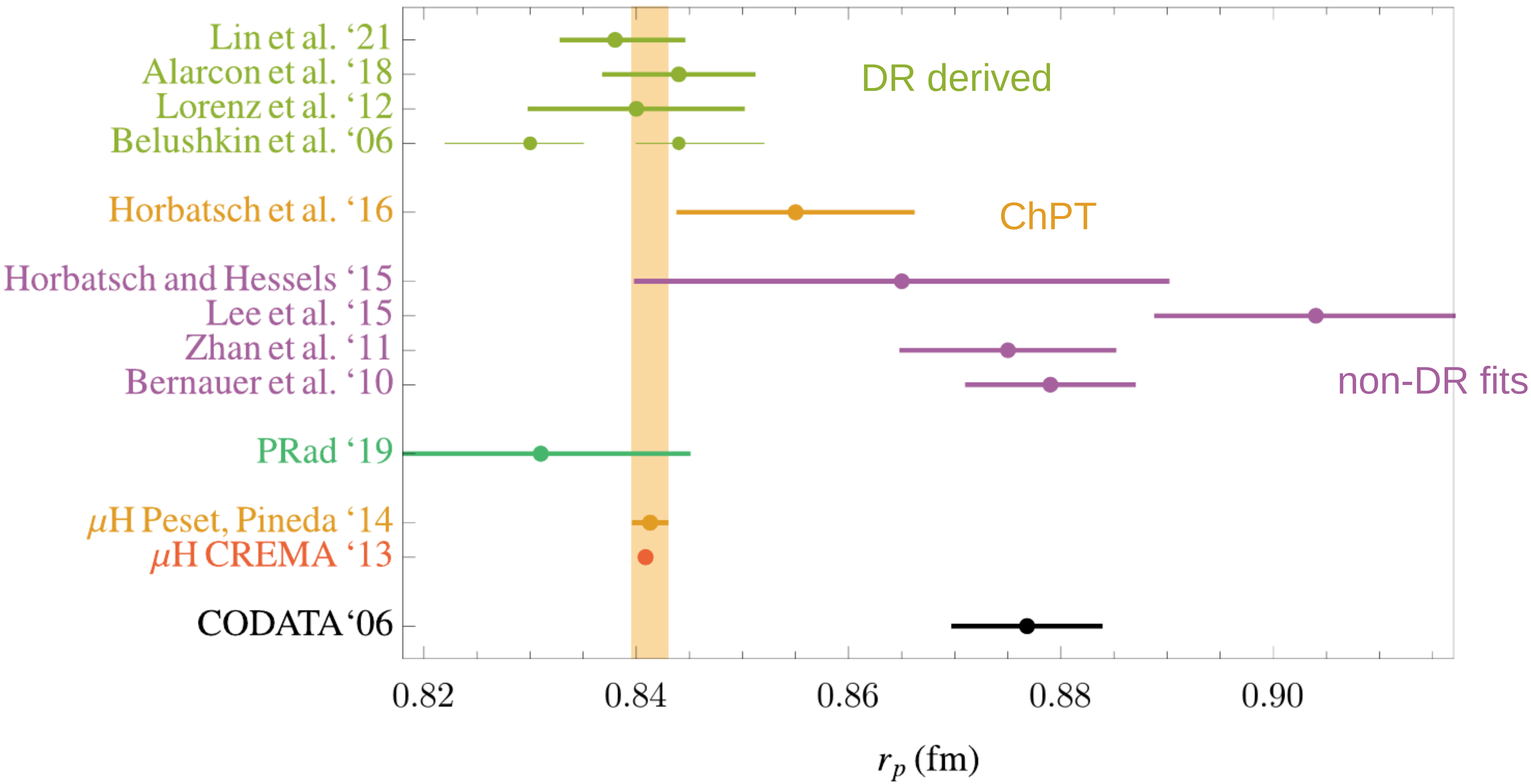
borrowed from Christian Dreisbach (TUM)



Status of proton radius "puzzle"

- Determination of ep scattering

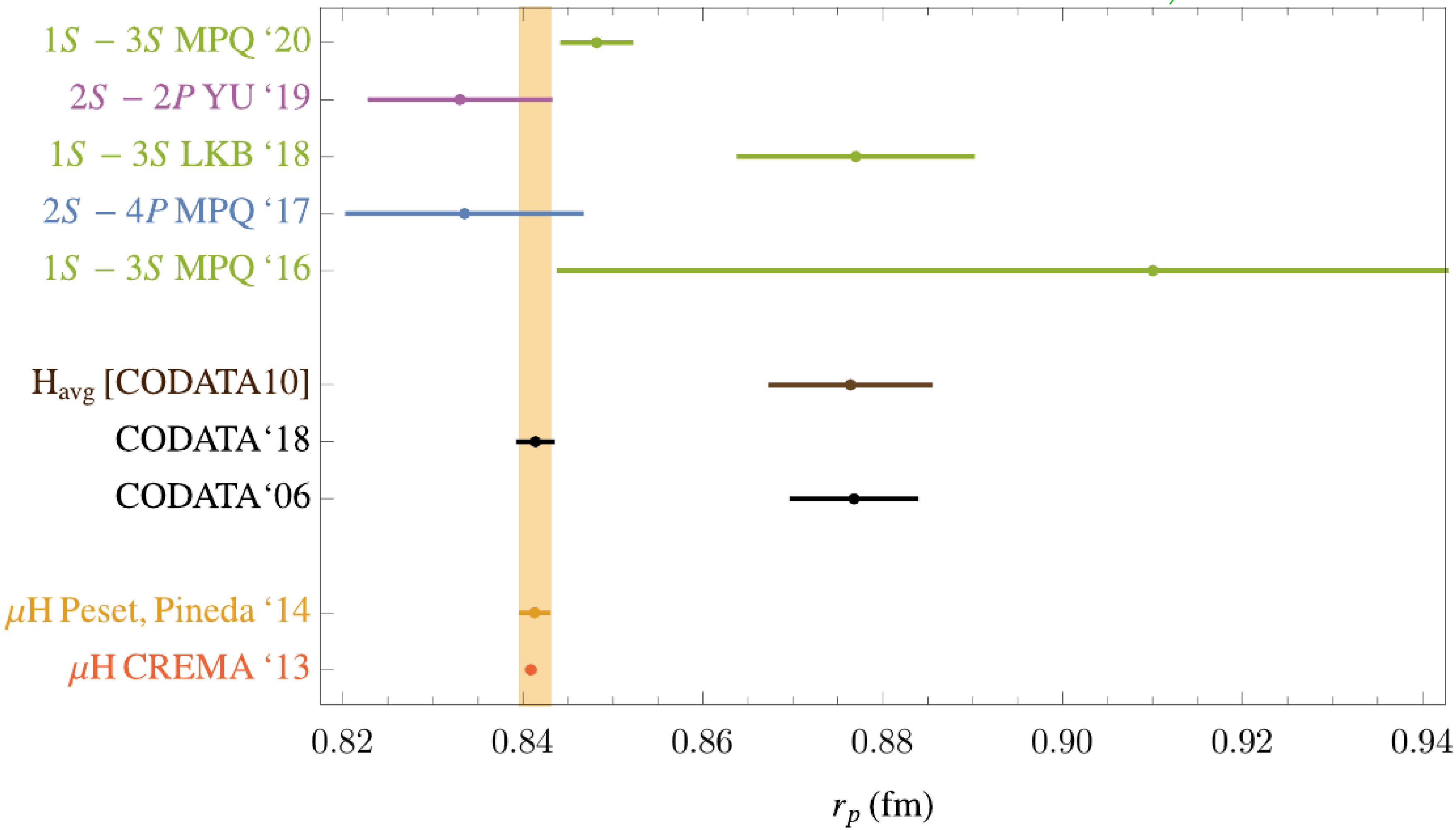
C. Peset, *et al.* 2106.00695



Status of proton radius "puzzle"

- Determination of hydrogen energy shift

C. Peset, *et al.* 2106.00695



From “puzzle” to precision

- Open discussion C. Peset, *et al.* 2106.00695

$$(\Delta E_L)_{\text{measured}} = E_1 + E_2 C(r_p^2) + \mathcal{O}(m_r\alpha^6), \quad C(r_p^2) = c_1 + c_2 r_p^2 + \mathcal{O}(\alpha^2)$$

δ_{TPE} ecoded in E_1, E_2, and $C \sim \mathcal{O}(m_l)$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{measured}} = \frac{d\sigma_{\text{Mott}}}{d\Omega}\frac{1}{1+\tau_p}\left(G_E^2 + \frac{\tau_p}{\varepsilon}G_M^2\right)(1+\delta_{\text{TPE}}) + \mathcal{O}(\alpha^2)$$

$\langle r_p^2\rangle, \delta_{\text{TPE}}$, higher moments($\langle r_p^n\rangle$) and polarizabilities

interwinded together when goes to the higher order corrections

Precision is the object that really matters in proton charge radius problem.

Summary

- We updated previous DR analysis on ep scattering data in following aspects,
  - Including the unprecedented low-transfer-momentum data by PRad
  - Improved high precise $\pi\pi$ continuum
  - Improved uncertainty estimation
- DR analysis on NFFs data provide robust, steady and consistent r_p^E over decades.
- Newest regular hydrogen measurements and DR determination definitely agree with 'small' μH-CREMA proton radius. Should not call it puzzle anymore.

Thank you very much for your attention!

Proton charge radius

- In Breit frame (non-relativistic limit), $e^- + p \to e^- + p$

$$q = (0, \vec{q}), \;\; Q^2 = -q^2 = \vec{q}^{\,2} \geq 0$$

$$G_{\mathrm{E}}(q^2) = \int d^3\vec{r}\, \rho_{\mathrm{E}}(\vec{r})\, e^{-i\vec{q}\cdot\vec{r}}$$

Considering a spherical density,

$$G_{\mathrm{E}}(q^2) = 2\pi \int_0^\infty r^2 dr\, \rho_{\mathrm{E}}(r) \int_{-1}^{1} d\cos(\theta)\, e^{-i|\vec{q}|\, r\, \cos(\theta)}$$

Re-define the momentum dependence, $|\vec{q}| = Q$

$$G_{\mathrm{E}}(Q^2) = \frac{4\pi}{Q} \int_0^\infty r\rho_C(r) \sin(Qr)\, dr$$

Proton charge radius

Expanding at $Q^2 = 0$

$$G_{\mathrm{E}}(Q^2) = 4\pi \sum_{j=0}^{\infty} (-1)^n \frac{\left(Q^2\right)^n}{(2n+1)!} \int_0^{\infty} r^{2n+2} \rho_{\mathrm{E}}(r)\, dr$$

$$= G_{\mathrm{E}}(0) \sum_{n=0}^{\infty} (-1)^n \frac{\left\langle r^{2n} \right\rangle_{\mathrm{E}}}{(2n+1)!} \left(Q^2\right)^n$$

$$\left\langle r^{2n} \right\rangle_{\mathrm{E}} \equiv \frac{4\pi \int_0^{\infty} r^{2n+2} \rho_{\mathrm{E}}(r)\, dr}{4\pi \int_0^{\infty} r^2 \rho_{\mathrm{E}}(r)\, dr} = \frac{1}{G_{\mathrm{E}}(0)} 4\pi \int_0^{\infty} r^{2n+2} \rho_{\mathrm{E}}(r)\, dr \,, n \in \mathbb{N}$$

Comparing with the Taylor series of $G_{\mathrm{E}}(Q^2)$, centered at $Q^2 = 0$

$$G_{\mathrm{E}}(Q^2) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n G_{\mathrm{E}}}{d\left(Q^2\right)^n} \right|_{Q^2=0} \left(Q^2\right)^n$$

$$\left\langle r^{2n} \right\rangle_{\mathrm{E}} = (-1)^n \frac{(2n+1)!}{n!} \frac{1}{G_{\mathrm{E}}(0)} \left. \frac{d^n G_{\mathrm{E}}}{d(Q^2)^n} \right|_{Q^2=0} , n \in \mathbb{N}$$

Parametrization of NFFs

- Our spectral functions of NFFs read

$$\mathrm{Im}\, F_i^s(t) = \mathrm{Im}\, F_i^{(s,K\bar{K})}(t) + \mathrm{Im}\, F_i^{(s,\rho\pi)}(t) + \sum_{V=\omega,\phi,s_1,\ldots} \pi a_i^V \delta(M_V^2 - t)\,,$$

$$\mathrm{Im}\, F_i^v(t) = \mathrm{Im}\, F_i^{(v,2\pi)}(t) + \sum_{V=v_1,\ldots} \pi a_i^V \delta(M_V^2 - t)\,, \quad i = 1,2\,.$$

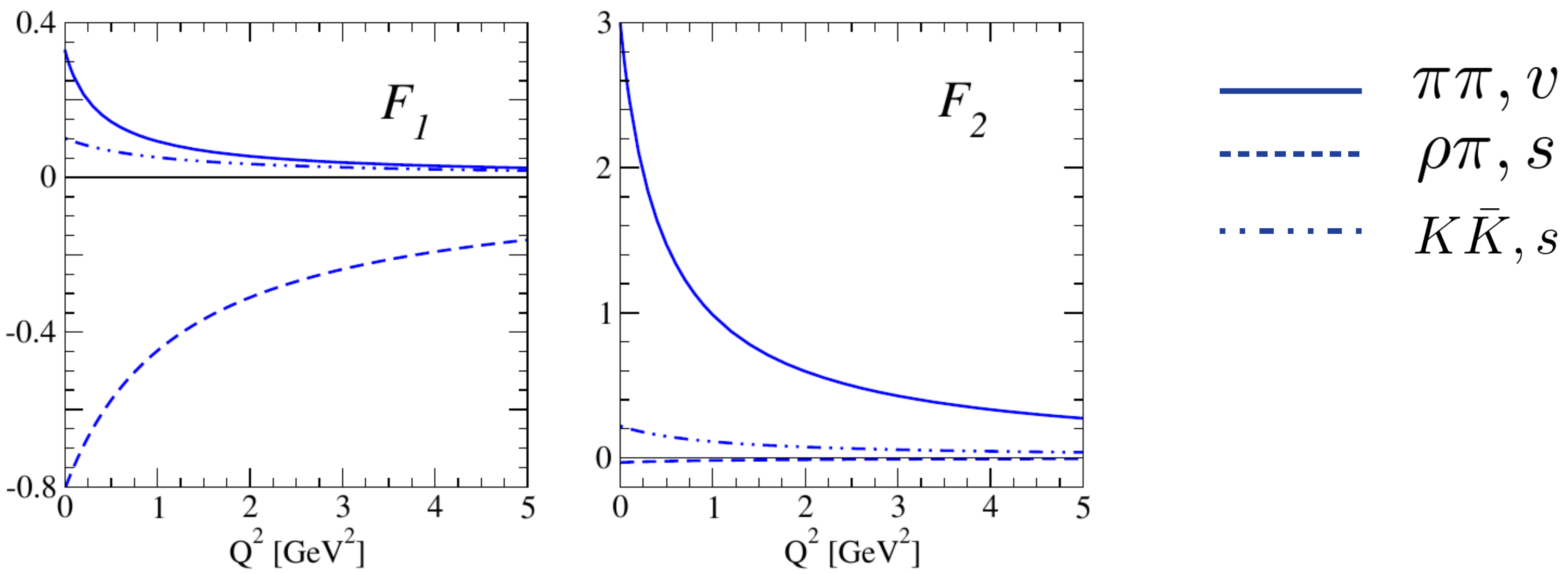
- $\mathrm{Im}\, F_i^{(s,K\bar{K})}(t)$, $\mathrm{Im}\, F_i^{(s,\rho\pi)}(t)$, $\pi a_i^V \delta(M_V^2 - t)$

  can be converted into (after DR integral)

$$F_i(t) = \frac{a_i^V}{M_V^2 - t}$$

Parametrization of NFFs

- Continuum contribution (not fitted to data)
  - M. Hoferichter, *et al.* EPJA52, 331(2016)
    $\pi\pi$ based on very precise analysis of pion-nucleon scattering
  - H.-W. Hammer, *et al.* PRC60, 045205(1999)
    $K\bar{K}$ from an analytic continuation of kaon-nucleon scattering data
  - U.-G.Mesißner, *et al.* PLB633, 507(2006)
    $\rho\pi$ from investigation of the Bonn-Jülich N-N interaction model



Two-photon-exchange correction

- Soft-photon approximation

$$\frac{d\sigma_{\rm corr}}{d\Omega} = \frac{d\sigma_{1\gamma}}{d\Omega}(1+\delta_{2\gamma}+\ldots)\ ,\ \delta_{2\gamma} \underbrace{\approx}_{\mathcal{O}(\alpha)} \frac{2{\rm Re}(\mathcal{M}^\dagger_{1\gamma}\mathcal{M}_{2\gamma})}{|\mathcal{M}_{1\gamma}|^2}$$

- One-gamma amplitude

$$\mathcal{M}_{1\gamma} = -\frac{e^2}{q^2}\bar{u}_e(p_3)\gamma_\mu u_e(p_1)\bar{u}_N(p_4)\Gamma^\nu u_N(p_2)$$

- Two-gamma amplitude

$$\mathcal{M}^{\rm box}_{2\gamma} = -ie^4\int\frac{d^4k}{(2\pi)^4}L^{\rm box}_{\mu\nu}(H^{\mu\nu}_N + H^{\mu\nu}_\Delta)D(k)D(q-k)$$

$$L^{\rm box}_{\mu\nu} = \bar{u}_e(p_3)\gamma_\mu S_F(p_1-k,m_e)\gamma_\nu u_e(p_1)$$

$$\begin{aligned}H^{\mu\nu}_N &= \bar{u}_N(p_4)\Gamma^\mu(q-k)S_F(p_2+k,m_N)\Gamma^\nu(k)u_N(p_2)\\ H^{\mu\nu}_\Delta &= \bar{u}_N(p_4)(p_4)\Gamma^{\mu\alpha}_{\gamma\Delta\to N}(p_2+k,q-k)S_{\alpha\beta}\\ &\quad\times(p_2+k)\Gamma^{\beta\nu}_{\gamma N\to\Delta}(p_2+k,k)u_N(p_2),\end{aligned}$$

Values of proton charge radius

- Historical DR determination

Ref.	r_E^p [fm]	r_M^p [fm]
Hohler:1976ax	0.836 ± 0.025	0.843 ± 0.025
Mergell:1995bf	0.847 ± 0.008	0.836 ± 0.008
Hammer:2003ai	0.848^*	0.857^*
Belushkin:2006qa	$0.844^{+0.008}_{-0.004}$	0.854 ± 0.005
Lorenz:2012tm	0.84 ± 0.01	$0.86^{+0.02}_{-0.03}$
Lorenz:2014yda	$0.840^{+0.015}_{-0.012}$	$0.848^{+0.06}_{-0.05}$
Lin:2021umk	$0.838^{+0.005}_{-0.004}{}^{+0.004}_{-0.003}$	$0.847 \pm 0.004 \pm 0.004$

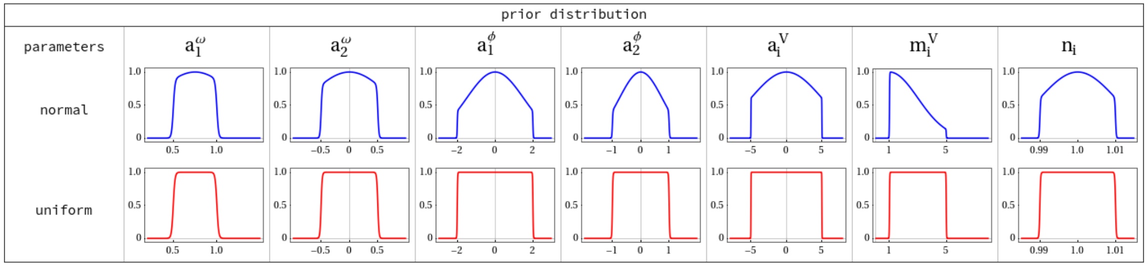
Bootstrap vs Bayesian in PRad fits

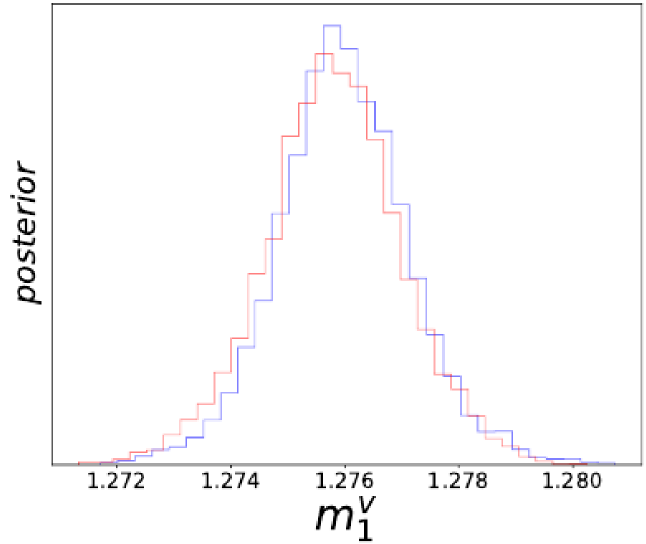
- Bayesian theorem

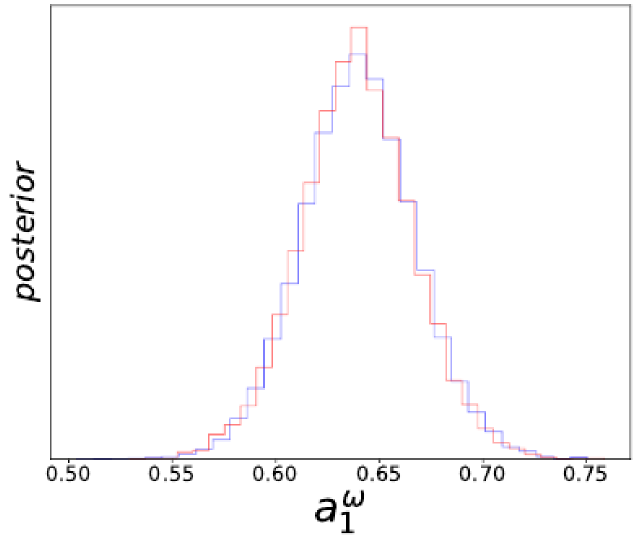
$$P(\text{parameters}|\text{data}) = \frac{P(\text{parameters})P(\text{data}|\text{parameters})}{P(\text{data})}$$

$$\text{posterior} \sim \text{prior} \times \text{likelihood}$$

- Bayesian analysis



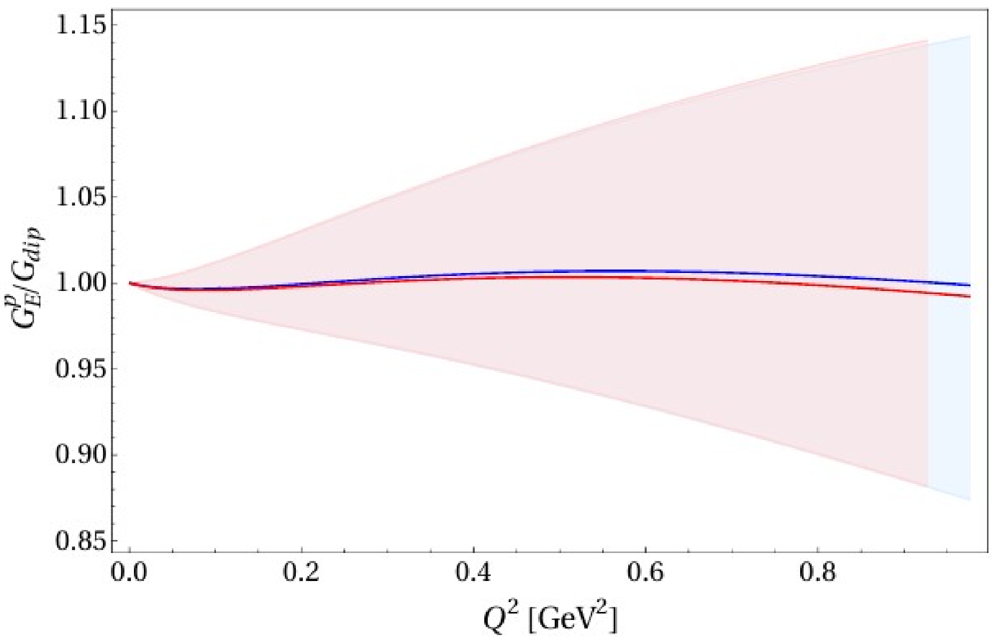


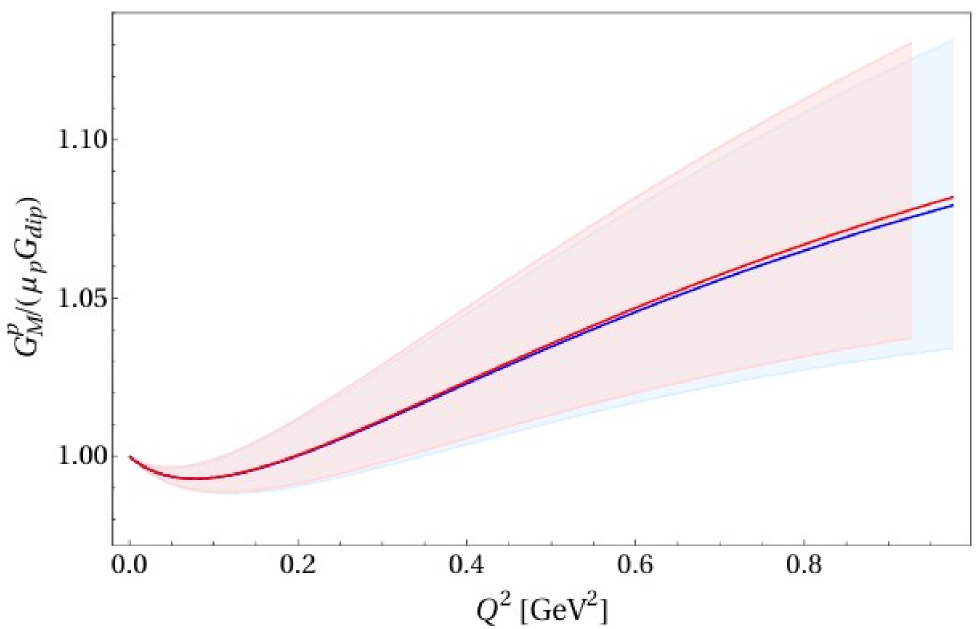


Bootstrap vs Bayesian in PRad fits

- Comparing with bootstrap sampling

Method	r_E^p [fm]	r_M^p [fm]
Bayesian normal	0.828 ± 0.011	0.843 ± 0.004
Bayesian uniform	0.828 ± 0.011	0.843 ± 0.004
Bootstrap	0.828 ± 0.012	0.843 ± 0.005





Parameters of best fit

- Best fit "6s+4v"

V_s	m_V	a_1^V	a_2^V	V_v	m_V	a_1^V	a_2^V
ω	0.7830	0.6893	0.0431	v_1	1.1222	1.0414	-0.6239
ϕ	1.0190	-0.0281	-0.4705	v_2	1.5147	-4.0062	-3.0365
s_1	1.8267	0.3768	0.5590	v_3	1.8062	4.8533	2.1897
s_2	4.0020	-1.2786	-4.882	v_4	2.2543	-2.0208	-0.0438
s_3	4.0713	1.8028	4.0681				
s_4	4.3075	-0.6576	0.4944				

n1	0.9965	n2	1.0061	n3	1.0028	n4	1.0010	n5	1.0035	n6	0.9914
n7	0.9982	n8	0.9929	n9	1.0076	n10	1.0000	n11	1.0000	n12	1.0037
n13	1.0030	n14	1.0044	n15	1.0055	n16	1.0027	n17	1.0048	n18	1.0013
n19	0.9995	n20	1.0029	n21	0.9977	n22	0.9905	n23	0.9985	n24	1.0100
n25	1.0080	n26	1.0069	n27	0.9999	n28	1.0100	n29	1.0066	n30	0.9999
n31	1.0100	ñ1	0.9989	ñ2	1.0059						